AMC 10 / 12 BASIC NOTES

Word Problems

Work Output
- Traveling Word Problems
- Working Word Problems

Ratios
- Mixture Word Problems
- Age Problems

Money
- Store-cost Problems
- Price/Pound Problems

Interest Problems
- Simple
- Compound

Interacting Word Problems

Algebra

Quadratic Formula Implementation

Implementing formulas ($a^2-b^2=(a+b)(a-b)$, etc.)

Magic Shapes

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- 2 sets
- 3 sets
- Multiple problems (either 3 or 2 sets)
- Range Problems

Geometry

Plane Geometry
- Solving for area (Different Question Types)
- Properties of common shapes
- Using Pythagorean Theorem
- Angles Problems
- Similarity
- Special Triangles
- Trigonometric Problems
- Circle Geometry

Solid Geometry
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Analytic Geometry
- Coordinate Plane Basics
- Reflection, Rotation, Translation

Number Theory

Modular Arithmetic
- Apply strategies and formulas
- Repeating Group Problems

Properties of numbers
- Evens and Odds
- GCF and LCM
- Primes
- Divisors
- Divisibility Rules

Game Problems:

Game Problems are unique in that there are many different types and significant variation from problem to problem. That being said, I will summarize some interesting game problems I have encountered.

There are a variety of common strategies that are employed to solve game problems. I will list a few hear along with a short explanation:

1. Case Bashing (This is a general strategy often employed in more basic problems)
    a. Case bashing involves working backwards and analyzing "base cases". Cases which result in a win for the first player choosing are called **N positions** while cases which result in a win for the second player choosing are called **P positions**.
    b. If the first player can change a position such that it becomes a P position, then that original position is an N position.
    c. If the first player can only change a position to N positions, then that original position is a P position.
2. Symmetry (This is very helpful in more complicated geometric problems):
    a. Sometimes, symmetrical strategies involve reflecting the other players move or producing a symmetrical plane and then reflecting the other players move.
        i. Another strategy involves placing your piece such that it is the conjugate of the prior players. (this is much rarer than regular symmetry)\
3. Nim Game: (Many games can be transformed into a nim environment. Once the game is transformed, it is much easier to solve.)
    a. Nim games can be solved by (1) case bashing or (2) finding a concrete reasoning (deeper than case bashing but not needed).
    b. NIM positions with a total XOR of 0 are P positions, otherwise they are N
        i. This can be employed within a lot of problems
        ii. The strategy is to produce positions with XOR = 0 whenever you have an N position

Problem 1:

Alphonse and Beryl play the following game. Two positive integers m and n are written on the board. On each turn, a player selects one of the numbers on the board, erases it, and writes in its place any positive divisor of this number as long as it is different from any of the numbers previously written on the board. For example, if 10 and 17 are written on the board, a player can erase 10 and write 2 in its place. The player who cannot make a move loses. Alphonse goes first.

(a) Suppose $m = 2^{40}$ and $n = 3^{51}$. Determine which player is always able to win the game and explain the winning strategy.

(b) Suppose $m = 2^{40}$ and $n = 2^{51}$. Determine which player is always able to win the game and explain the winning strategy.

(a) For this, it's a good idea to work from end to start. It's a good idea to consider the powers of m and n rather than the exact values. Let x,y=the degree of m and n respectively. Now, we can work recursively start from a known winning/losing position x=0,y=1 is a losing position

x,y=2,2 is a losing position

x=2, y>2 is a winning position

x,y=3,3 is a losing position (because, if the person makes x,y>=2, then the other person will just make x=y)

x=y will always be a losing position because the other person can always reinforce x=y until x,y=2,2 or until x=(0,1),y=(1,0).

Thus, Alphonse has the winning strategy because he can make x=y=40 in the first move and can enforce x=y for the rest of the game until Beryl makes x < 2 or y < 2.

(b) For this, the base is the same, so there is a risk of repetition (because, once a number has been made, it can never be made again). We can, however, emply a similar recursive approach.

x!=0,y=1 is a winning position

x=2, y=3 is a losing position but y>3 is a winning position

x=3, y=4 is a winning position but y>4 is a losing position

x=4, y=5 is a losing position but y>5 is a winning position

…

x=40, y=41 is a losing position but y>41 is a winning position.

Hence, alphonse wins

Word Problems

Traveling word problems:

Primary Formula: distance=time*speed

Types of problems:

1. Two-part and average speed problems
2. Meeting problems
3. Catch-up problems
4. Ratio Problems
5. Circular track problems
6. Acceleration Problems

Two part and average speed problems:

1. Target is total distance or the distance of one section
2. Target is average speed or average speed for a section
3. Target is the total time or time for a section (Similar to distance)

Problem 2 (average speed): Total journey is 600km. A bus travels the first 120 km of the trip at a speed of 30km per hour. What should be the speed for the next 480 km so that the average speed is 60 km/hour?

Formula: $averageSpeed = \frac{totalDistance}{totalTime}$

Strategy: Determine the values that contribute to the totalDistance and total Time variables and develop an equation;

$t1 = \frac{120}{30} = 4\ hours$

$t2 = 480/s2$

$60 = 600/(4 + \frac{480}{s2})$

$\frac{480}{s2} = 6$

$s = 80$

Meeting Problems:

Problem 2: (meeting): Two towns are 400 miles apart. A car leaves the first town traveling toward the second town at 55 mph. An hour later, a second car leaves the other town and heads toward the first town at 65 mph. How long will it take for the two cars to meet?

Strategy:

1. Use relative speed to solve (implementing formula)
2. Use algebra to create equation(s) based upon the d=s*t principle;

Method1:

In order to use the formula, the time of travel must be equal for both vehicles. To make the time traveled equal, we must account for car A's hour "headstart".

The new distance between the cars becomes 400-55*1=345.

Plugging these values into the equation, (55+65)t=345; t=345/120=23/8 hours;

Method2:

55(t+1)+65t=400;

t=345/120=23/8 hours;

Catch Up Problems (Very similar to Intersecting Problems):

Problem (catch-up): The Hill family and the Platter family are going on a road trip. The Hills left three hours before the Platters but the Platters drove an average of 15 mph faster. If takes the family 13 hours to catch up, how fast are the hills driving?

Strategy:

1: If possible use the common formula to solve

2: If the first option is impossible, create 2 equations using d=t*s

Method1:

Because the time of travel is different, we must account for it before plugging in the values into the formula.

Let Hill's speed=s; Let Platter's speed = s+15;

Before the Platters travel, the Hills travel for three hours at s, such that, once the Platters started, the Hills were 3s

ahead. Knowing these values, we can now use the relative speed formula;
(s+15-s)13=3s
3s=195;
s=65 km/hour

<u>Method2:</u>

$d = (s + 15) * 13$
$d = 16s$
$(s + 15) * 13 = s * 16$
$s = 65\ km/hour$

<u>Ratio Problems:</u>

Problem (combine): A, B, and C are racing on a 100 meter track. When A finished the race, B was 20 meters from the finish line. When B finished the race, C was 10 meters from the finish line. How many meters had C traveled when A finished the race?

Strategy: Combine the ratios given by making B's distance equal in both ratios;
Ratio A:B = 100:80; Ratio B:C = 100:90 => 80:72
Ratio A:B:C=100:80:72;
C had traveled 72 meters when A finished the race.

<u>Circular Track Problems:</u>

Types of problems:
1. Next meeting point (meeting, catch-up)
2. The next time they will both be on the starting point (meeting, catch-up)
3. Number of intersections

Next meeting point:

Problem1 (same direction): Annie and Bonnie are running laps around a 400-meter oval track. They started together, but Annie has pulled ahead because she runs 25% faster than Bonnie. How many laps will Annie have run when she first passes Bonnie?

<u>Strategy:</u>
1. Relative Speed
2. Ratios/Logics

Relative Speed:

You can imagine this question as a catch-up problem in which the two cars are separated by 400 meters. The car which starts behind is driving 25% faster.
Using this interpretation, we can plug in the values into the formula:
(5/4s-s)t=400; => 1/4 * (st) = 400; Thus, 5/4 * st = 5 * 400 = 2000 meters = 5 laps;

Ratio:

Every lap completed by driver B, the driver A completes 25% of a new lap such that, after 4 laps made by the driver B, driver A drives one additional lap and is at the same point as driver B.
Thus, it takes 5 laps for driver A to pass driver B;

Problem2 (opposite direction and same direction): Albert and Bob start at the same point to run clockwise around a circular track with 600 meters, and they meet every 12 minutes. If they start at the same point to run in the opposite direction, they meet every 4 minutes. How many minutes do Albert and Bob need to run a lap?

Strategy: Relative Speed

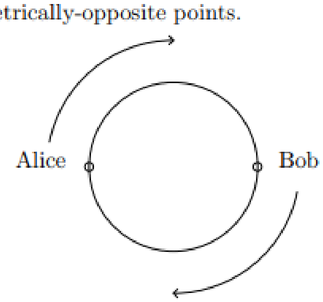
You can use both formulas to create two equations;

1. (s1-s2)12=600; s1-s2=50
2. (s1+s2)4=600; s1+s2=150

Albert takes 600/50 = 12 mins
Bob takes 600/100=6 mins

Problem 3:

B1 Alice and Bob run in the clockwise direction around a circular track, each running at a constant speed. Alice can complete a lap in t seconds, and Bob can complete a lap in 60 seconds. They start at diametrically-opposite points.



When they meet for the first time, Alice has completed exactly 30 laps. Determine all possible values of t.

Strategy: Logics

Alice finishes exactly 30 laps meaning that she must end up at the same point she started on. This means that Bob must have either completed 30.5 or 29.5 laps. We can use these two values to determine the two possible values for t. 30t=30.5*60; || 30t=29.5*60; t=61,59;

Next time they will both be on the starting point:

Problem: Two balls A and B rotate along a circular track. Ball A makes 2 full rotations in 26 minutes. Ball B makes 5 full rotations in 35 minutes. If they start rotating now from the same point, when will they be at the same starting point again?

Strategy: Determine the time per lap of each ball and calculate the LCM. *(Time it takes for both entities to get back to the starting position = LCM(A’s time/lap, B’s time/lap))*

Ball A = 26/2 = 13 minutes/lap
Ball B = 35/5 = 7 minutes/lap
LCM = 91 minutes;

Number of times of intersection

Problem: Binh and Duong are running on a circular track. They start at the same place, but run in opposite directions. Binh runs at 15 km/h and Duong runs at 12 km/h. After they have each made several laps they arrive simultaneously at their starting place, at which time they stop. How many times will they have met on the track during their run, not including the start and the finish?

Strategy: Use the speeds of Binh and Duong to calculate the ratio between their time/lap. Once these values have been calculated, find their LCM and determine how many laps each person needed to travel. Now add these values.

Let D = the total distance of the circular track.
Binh's time/lap = D/15; Duong's time/lap = D/12; Binh:Duong = 12:15 = 4:5; LCM(4,5)=20;
Binh's lap quantity = 20/4=5; Duong's lap quantity = 20/5=4;
Number of intersections = 5+4=9; Discounting the final intersection = 8;

Advanced Contest Problems:

Problem 1: Two joggers each run at their own constant speed and in opposite directions from one another around an oval track. They meet every 36 seconds. The first jogger completes one lap of the track in a time that, when measured in seconds, is a number (not necessarily an integer) between 80 and 100. The second jogger completes one lap of the track in a time, t seconds, where t is a positive integer. The product of the smallest and largest possible integer values of t is what?

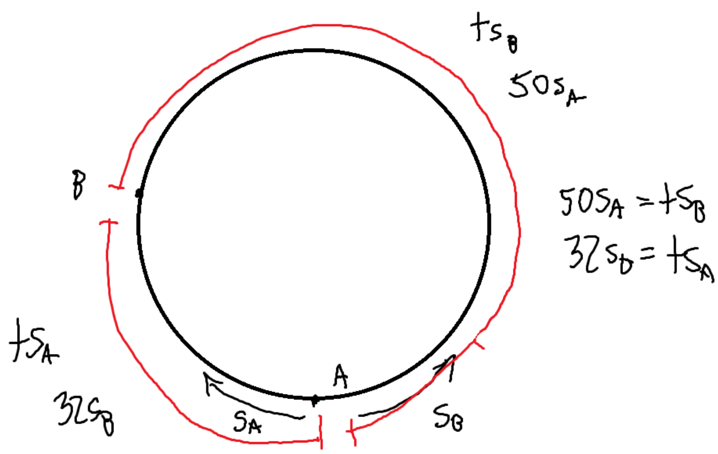
Strategy: Assume the values of each of the joggers' time per laps and create an equation based on the time-to-meet condition. After, substitute the two extremes for the first joggers' time and solve for the time of the second jogger.

Let t1= the first jogger's time per lap; and t2=the second "", where the target is t2;
Such that, $\frac{1}{t1} + \frac{1}{t2} = \frac{1}{36}$; We can make this equation more "inputtable" => $\frac{1}{t2} = \frac{1}{36} - \frac{1}{t1}$;
Let t1=80; If this is true => t2=floor(720/11) ; = 65
Let t1=100; If this is true => t2=ceil(225/4); = 57
Their product => 57*65=3705;

Problem 2: Abe and Beth ran exactly one full circular lap on the school track (they both ran the exact same distance). They both started from point A on the track and both started at the same time, but were running in opposite directions as indicated on the diagram. Beth finished 32 seconds after they passed each other (at point B), and Abe finished 50 seconds after they passed each other. Each ran at constant speed. How long (in seconds) did Beth take to run the lap? (Hint: it does not matter how long the lap was).

Let the respective speeds of Abe and Beth be Sa and Sb.
Let t be the time both Abe and Beth took to intercept.

Drawing the following diagram:



We can now solve the equations: 50Sa=tSb & 32Sb = tSa for the target, t.

Thus t=40 and the total time it takes for B to complete a full lap is 72 seconds.

Problem 3:

Three runners start running simultaneously from the same point on a 500-meter circular track. They each run clockwise around the course maintaining constant speeds of 4.4, 4.8, and 5.0 meters per second. The runners stop once they are all together again somewhere on the circular course. How many seconds do the runners run?

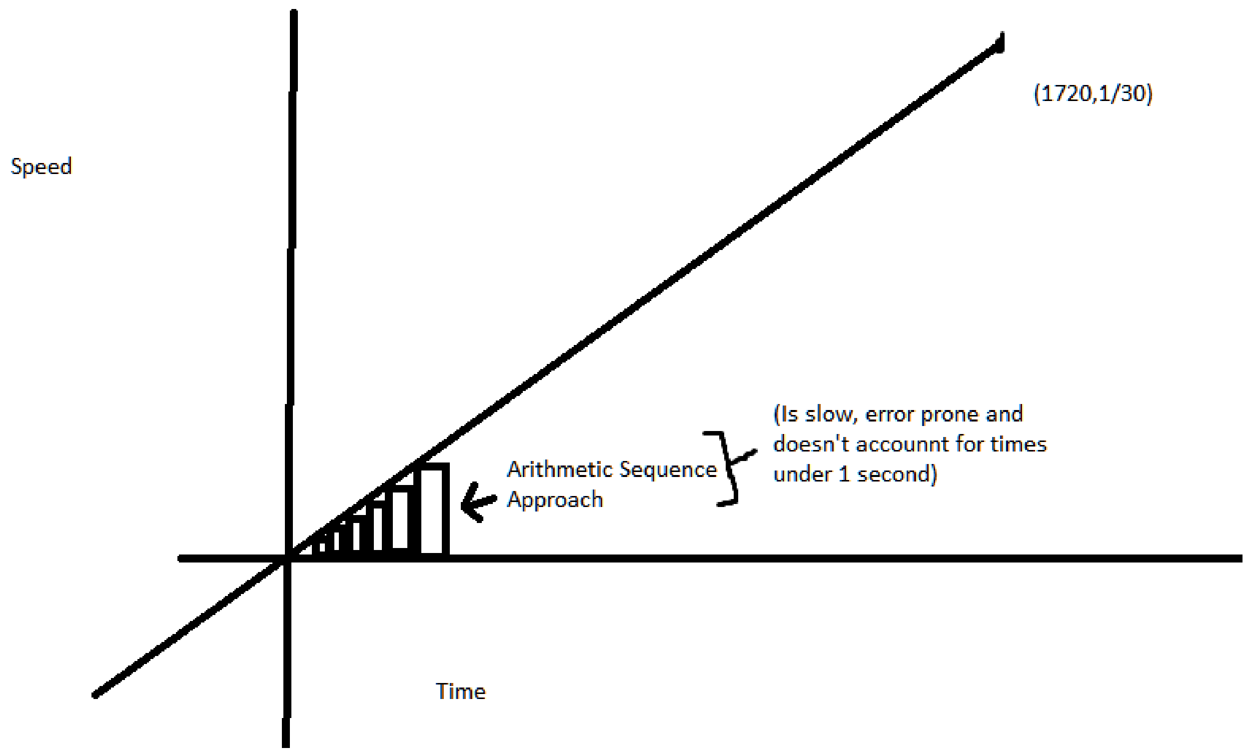
(A) 1,000 **(B)** 1,250 **(C)** 2,500 **(D)** 5,000 **(E)** 10,000

<u>Acceleration Problems:</u>

8. Nimoy drives a car starting at a speed of 0 km/h and accelerating at a constant rate, and driving along the circular road around CERN (the Large Hadron Collider). When he finished one circle he reached a speed of 120 km/h. It took him exactly 1720 seconds to reach that speed. Find the radius of the circle in km correct to one decimal.

Problem 1:

This problem is difficult to visualize, particularly if you try and use the a conventional method like creating an arithmetic sequence. This “artihmetic sequence” method is also inaccurate and may lead to a calculation error. Alternatively, for acceleration problems, we should create a graph that displays the speed and time on the y and x axises respectively as shown:



Thus, we can see that the answer is the: 2*pi*r = speed * the final time * ½ = 1720*1/30*½; Thus, r=4.6

Working Problems (Extremely Similar to Traveling Problems):

There are three common types of problems:
1. Solving for the combined
2. Solving for the individual
3. Solving for either, but forces work against each other
4. Challenge Problems

Solving for the combined:

Problem: If it takes Felicia 4 hours to paint a room and her daughter Katy 12 hours to paint the same room, how long will it take if they work together?

Target: Individual combined time;
Strategy: Use formula determining all necessary variables and solving; (Assuming the total workload is very common);

Formula: $totalTime = \frac{totalWorkload}{totalSpeed}$

let room=$1m^3$

$s1 = \frac{room}{time} = \frac{1}{4}$

$s2 = \frac{1}{12}$

$\frac{1}{\frac{1}{4}+\frac{1}{12}} = totalTime$

4*t/12=1; $4t * \frac{t}{12} = 1$

$t = 3\ hours$

<u>Solving for individual:</u>

Problem: Karl can clean his room in 3 hours. If his little sister Kua helps, they will clean it in 2.4 hours. How long does it take Kya to clean by herself?

Target: Individual time;
Strategy: Use formula and create an equation;
Formula: $totalTime = \frac{totalWorkload}{totalSpeed}$
let room=$1m^3$

$s1 = \frac{1}{3}$

$s2 = \frac{1}{t}$

$\frac{1}{\frac{1}{3}+\frac{1}{t}} = 2.4$

2.4t+7.2=3t;
x=12;

Solving for either but having forces work against each other:

Problem: A sink can be filled by a pipe in 5 minutes, but it takes 7 minutes to drain a full sink. If both the pipe and the drain are open, how long will it take to fill the sink?

Target=total time;
Strategy: Use formula and substitute values;
Formula (s1-s2)*t=workload total; //similar to (s1-s2)t=d;
(1/5-1/7)t=1;
2/35=1;
t=35/2 minutes;

Solving challenge work-rate problems:

Problem 1:

<u>Statistics Problems:</u>

Generally there are two ways to solve statistics problems:

1. Using Algebra (More generalized and easy to use)
2. Using Difference Method (Faster but more difficult to use)

Problem 1: (Basic Statistic Question): Ruby mailed three packages to a friend. The mean weight of the packages was 85 ounces. When Ruby sent a fourth package, the mean weight increased by 2 ounces. How many ounces did the final package weigh?

Method 1: Algebra:

(85*3+x)/4=87; => x=87*4-85*3=93;

Method 2: Difference Method:

In order to increase the mean by two, the total weight must increase by eight more than the mean. Thus, the fourth package has a weight of 8+85=93;

Problem 2: (Basic Statistic Question): The mean of three consecutive terms in an arithmetic sequence is 10, and the mean of their squares is 394. What is the largest of the three original terms?

If the mean of an arithmetic sequence has a difference d, then the terms can be represented as shown below:

10-d,10,10+d

Squaring the terms, we get: (10-d)^2+(10+d)^2+100=394*3; => 2d^2=982; => d=21;
Largest term = 10+21 = 31;

Problem 3: (Range-Based Statistic): After taking three of the four exams in history class, Srinivasa has an average exam score of 66 points. If the fourth exam counts twice as much as the other exams, what are the fewest points Srinivasa can score on the fourth exam to pass the course with an overall exam average of at least 70 points?

<u>Algebra Method:</u>

(3*66+2x)/5>=70; x>=(350-66*3)/2=76;

<u>Difference Method:</u>

We can assume the average to be exactly 70 and, if there happens to be a decimal result, we may simply round up. We can view this problem as trying to preserve balance. With three scores of 66, the balance on one side is -4*3=-12; To sustain the balance, we must have a total of 12 greater than the average split between two exams. Thus, the lowest score is 76.

Problem 4: (Constraint-Based Statistic): The mean median and unique mode of six positive integers are 8,7,3, respectively. What is the maximum possible value for the range of the six numbers?

Strategy: Use the constraints to construct a computable group of cases and simply test each of them for the largest.

There are two possibilities for the first three/four integers based on the info given: (1,3,3,11,12,18), (3,3,6,8,9,19).
Range 1 = 17;
Range 2= 16;

Problem 5: (Constraint-Based Statistic): Six different positive integers add to 66. If one of them is the mean and another is the range, what is the largest possible number in the set?

Mean = 66/6=11; Let numbers = a,b,c,11,d-a,d; In ascending order. In order to find the maximum integer in the set, we must make the first three integers as small as possible: (1,1,2,11,24.25)

<u>Problem 6: (Counting-Based Statistics + Advanced):</u> Consider positive integers a<=b<=c<=d<=e. There are N lists a,b,c,d,e with a mean of 2023 and a median of 2023, in which the integer 2023 appears more than once, and in which no other integer appears more than once. What is the sum of the digits of N?

If 2023 exists more than 2 times in the list N, then the list will consist of all 2023's because the median must be 2023. Thus, the only case to consider is when 2023 occurs twice.

Case 1: (c=2023, b=2023)

In this case, a+d+e = 2023 * 3, and a<2023<d<e

The minimum value for d is 2024 (+1). So e can be from 2025 (+2) to 4044 (+2021). [2020]
The next value for d is 2025 (+2). So e can be from 2026 (+3) to 4043 (+2020) [2018]
…
The maximum value for d occurs when a is minimized (a=1 || -2022). In this case, the maximum value for d is 1010 and e will be 1012 or 1011. [2]

Counting the arithmetic sequence we get: 1011*1010.

Case 2: (c=2023, d=2023)

In this case, a+b+e=2023*3, and a < b < 2023 < e

Counting Method 1: (Arithmetic sequenece counting)

The minimum value for a is 1 and b can then span from 2 to 2022 [2021 cases]
The next value for a is 2 and b can then span from 3 to 2022 [2019 cases]
….
The maximum value for a is 2021 and b can only be 2022 [1 case]

Counting the arithmetic sequence: 2021*1011

Counting Method 2: (Combinatorics)

We are essentially choosing two distinct numbers from the set of numbers [1,2,3…2022] and placing them in increasing order. We can solve for the number of cases with 2C2022 = 2021*1011

Case 3: (everything is 2023) Only 1 occurrence of this case. (2023,2023,2023,2023,2023)

Solution: Case 1 + Case 2 + Case 3 = 1011*1010 + 2021 * 1011 + 1 = (A sum of 22)

Mixture Word Problems:

There is really only one basic type of mixture problem, just with different target values. There will only be two example questions in the category. The are two useful methods to solve:

1) Algebra (slower) - The image below, shows how equations are generally formulated.

Mixture word problem

Amount	Percent	Total

2) Difference Method (more efficient) - The difference method involves the use of ratios to deduce the result. Examples will be provided within both problems;

Problem 1: We have 80 liters of a 2% saline solution. How much of a 10% saline solution should we add to increase the salinity to 4%?

Strategy:
1. Solve using algebra

Algebra:
2%*80+10%*x=(80+x)*4%;
160-+10x=320+4x;
6x=160;
x=80/3 liters;

Problem 2: We have 10 gallons of a 26% alcohol solution and we need 15 gallons of an 18% alcohol solution. What % alcohol solution should we add to the 26% solution to get the desired solution?

Strategy:
1. Solve using algebra

10*26% + (15-10)*x=18%*15;
260 + 5x = 270;
x=2 liters;

Age Problems:

These problems are quite basic and don't vary extensively from question to question; thus, the single example;

Problem: A brother is twice as old as his younger brother. After 6 years ratio of their ages will be 3:4. What was the ratio of their ages two years ago?

Strategy: Use the information given to construct equation/s based on their age ratio. **(Remember to pay attention to the changes in present, past, and future).**

a/b=2/1; a+6/b+6=4/3;

a=2b => 6b+18=4b+24;
b=3,a=6;
6-2:3-2=4:1;

Money Problems:

Types of Problems:
1. Store-cost Problems
2. Price/Pound Problems
3. Interest Problems

Formulas:

Store-cost:
1. Selling Price=wholesale * (1+markup rate);
2. Final Selling Price = Regular selling price || Regular selling price * (1-discount rate);
3. Profit = Final Selling Price - Wholesale price;
4. Profit Rate = Profit/wholesale;

Price/Pound Problems:
5. Price per Pound= Total Price / Total Pounds;

Interest:
6. SI Formula S.I. = Principal × Rate × Time;
7. CI Formula C.I. = Principal (1 + Rate)^Time - Principal;

Store-cost Problems:

Problem 1: A store purchased two brands of monitors with a total cost of $500. The mark-up rate on the monitors are 50% and 40% respectively. The monitors are sold with 10% off, which yields 157 profit total. What are the costs of the two brands of monitors individually?

Strategy: Use formulas to create equations and solve for target values;

let monitor 1=x; monitor 2=y;
Formulas:
Selling Price = Wholesale*(1+markup);

Profit = Final - Wholesale;

Equations:
x+y=500;
RegPrice1=1.5x;
RegPrice2=1.4y;
Final=0.9(1.5x+1.4y)=1.35x+1.26y;
Profit=1.35x+1.26y-x-y=157;

Reduce
26x+26y=13000;
35x+26y=15700;
x=300,y=200;

Problem 2: A store makes $48 profit by selling a chair. If the store earns the same profit by selling 6 chairs with 10% off as by selling 9 chairs with $30 off. What is the purchase cost of one chair;

Strategy: Use formulas to create equations and solve for target value;

let wholesale=y;
let regularSellingPrice=x;

x-y=48;
6y*90%-6x=9y-30-9x;
5.4y-5x=9y-30-9x;
30=3.6y-3x;
144=-3y+3x;
174=0.6y;
y=290;
x=338;

Price/Pound Problems:

Problem: Find the selling price per pound of a chocolate mixture made from 5 pounds of raisins that sells for $13.00 per pound and 15 pounds of black chocolate that costs $21.00 per pound.

Strategy: Substitute to formula
Formula:Total Price / Total Pounds

Substitute:
5*13+15*21/20=65+315/20=380/20=19;

Interest Problems:

Types of Interest Problems:

1. Determining Simple Interest
2. Determining Compound Interest

Determining Simple Interest:

Problem 1: Robert deposits $3000 in State Bank of India for 3 years which earns him an interest of 8%. What is the amount he gets after 1 year, 2 years, and 3 years?

Strategy: Substitute values into formula:
Formula: S.I. = Principal × Rate × Time; //This amounts to the money gained through interest. If the target is the Final amount, you should add the
Principal to the result;
S.I=3000*0.08*1+3000=3240 after 1 year;
S.I=3000*0.08*2+3000=3480 after 2 years;
S.I=3720 after 3 years;

Problem 2: Richard deposited $5400 and got back $6000 after two years. Find the rate of interest.
Target: Rate of interest;
Strategy: Substitute values;
Formula: S.I. = Principal * rate * time;
6000-5400=5400 * rate * 2;
rate=600/10800=1/18 gain every year;

Determining Compound Interest:

Problem1: If you deposit $4000 into an account paying 6% annual interest for 5 years, how much money, in total, will you have after 5 years? *(Rounded to the nearest hundredth).*

Strategy: Substitute values into formula;
Formula: C.I. = Principal (1 + Rate)^Time - Principal;
// *this calculates the total gain, not the total value.*

C.I=4000(1+0.06)^5=$5352.90;

//Most of the time you need a calculator;

Problem 2: If you deposit $5000 into an account paying 6% annual interest compounded monthly, how many years until there is $8000 in the account? (To the nearest hundredth).

Strategy: Substitute values into formula;
Formula: C.I. = Principal (1 + Rate)^Time - Principal;
//Notice how the interest is compounded MONTHLY. This is very COMMON and must accounted for in the formula. The time variable is in years, so we must multiply the monthly compound by 12 and divide the rate by 12;

8000=5000(1+0.06/12)^12*t;
This can be solved using logarithms with a calculator;
8000=50001.005^12t
1.6=1.005^12t;
log(1.6)/log(1.005)=12t;
12t=94.2355;

t= Approximately 7.9 years;

Miscellaneous Problems:

Interaction Word Problems:

Problem 1: After a hockey game, each member of the losing team shook hands with each member of the winning team. Afterwards, each member of the winning team gave a fist-bump to each of her teammates. Each team has 20 players. If n handshakes occurred and m fist-bumps occurred, what is the value of $n + m$?

Strategy: Identify and separate when each individual in the group is shaking hands with the separate group and when a group shakes hands among itself.

Case 1: The group shakes hands with the other group:
$20*20=400$
Case 2: The group shakes hands among itself
$2C20=20*19/2=190$

Problem 2: In Jen's baseball league, each team plays exactly 6 games against each of the other teams in the league. If a total of 396 games are played, how many teams are in the league?

Strategy: Count the number of times two groups play each other and multiply by 6. (Be careful of over counting)

$2CN*6=396$; => $N(N-1)/2*6=396$; => $N(N-1)=132$; => $N=12$;

Algebra

Equations:

Convenient Systems are systems of equations that have a nice form that is slick and easy to solve. For example, some systems of equations are set up such that adding them all together can allow you to find the sum of all the variables. T

Symmetric Expressions and Advanced Factorization:

$a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b \dots + b^{n-1})$

$a^{2n+1}+b^{2n+1}=(a+b)(a^{2n}-a^{2n-1}*b \dots +b^{2n})$

$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$(a+b+c)^3 = a^3+b^3+c^3+3(ab^2+bc^2+ac^2+ba^2+ca^2+cb^2)+6abc$

More Polynomials: If the function of a linear equation has 2 roots, then all of the values of the coefficients and constants of the linear equation must be 0. We can extend this to any function of degree n: If the function has n+1 roots, all values of the coeffieicents and constants must be 0. **This idea helps us with proof problems involving algebraic manipulations.**

Squaring and cubing numbers: When an equation is riddled with square / cube roots a last resort often involves squaring or cubing both sides and hoping that the annoying square / cube roots are removed or at least presented more simply.

Problem1: Find the natural solutions (x,y) such that $x^2 + 3 = y(x + 2)$ **(**Algebra + Number Theory**)**

Method 1: (Generalized Quadratic Method)

x^2-yx+3-2y=0

Thus, the discrimint: y^2+8y-12 must equal a perfect square

Testing a few values, leads us to y^2+8y-12=36;

(y-4)(y+12)=0

y=4 is the only positive root.

x=5 is the only positive corresponding root.

Thus (x,y)=(5,4).

Method 2: (Divsibility)

y = (x^2+3)/(x+2)

y = [(x^2+2x)-2x+3] / (x+2)

y = x + (-2x+3)/(x+2)

y = x - [(-2x-4) + 7]/(x+2)

y = x - 2 + 7/(x+2)

x=5 is the only positive integer value that also results in y being a positive integer:

Thus, (x,y)=(5,4)

Problem 2: Find 5a+7b+9c if on zero a,b, and c satisfy the equations: (algebra manipulation)

ab = 2(a+b)

bc = 3(b+c)

cd = 4(c+a)

1=2(a+b)/ab

=> 1/2 = (1/b+1/a)

=> 1/3 = (1/b+1/c)

=> 1/4 = (1/a+1/c)

13/24=(1/a+1/b+1/c)

Then we find a,b,c and sub into the target

Problem 3: Prove that the equation $4a^2+4a=b^2+b$ has no positive integer solutions a,b: (algebra + number theory)

$4a^2+4a-(b^2+b)=0$
$a = [-4+\sqrt{16+16b^2+16b}]/4$
$a =[-1+\sqrt{b^2+b+1}]/2$
$b^2<(b^2+b+1)<(b+1)^2$
Thus, b^2+b+1 is not a perfect square and a can never be an integer.
The method used above is common and called "square sandwich".

<u>Polynomials:</u>

<u>Multiplying and dividing polynomials:</u>

1. **Long Multiplication Method**:

$$
\begin{array}{rrrrrr}
 & & 4x^3 & & -3x & +6 \\
\times & & & 2x^2 & -4x & -3 \\
\hline
 & & -12x^3 & & +9x & -18 \\
 & -16x^4 & & +12x^2 & -24x & \\
8x^5 & & -6x^3 & +12x^2 & & \\
\hline
8x^5 & -16x^4 & -18x^3 & +24x^2 & -15x & -18
\end{array}
$$

2. **Long Division Method**:

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-3 & x^3 - 2x^2 + 0x - 4 \\
 & \underline{x^3 - 3x^2} \\
 & \quad +x^2 + 0x \\
 & \quad \underline{+x^2 - 3x} \\
 & \qquad +3x - 4
\end{array}
$$

3. **Synthetic Divison**: Whenever a polynomial is being divided by a linear polynomial one can invoke the following strategy:

$$
\begin{array}{r|l}
 & x + 6 \\
\hline
x-1 & x^2+5x+6 \\
 & \underline{x^2 - x} \\
 & \quad 6x+6 \\
 & \quad \underline{6x-6} \\
 & \qquad 12
\end{array}
\qquad
\begin{array}{r|rrr}
1 & 1 & 5 & 6 \\
 & & 1 & 6 \\
\hline
 & 1 & 6 & 12
\end{array}
$$

<u>Finding Roots of Polynomials:</u>

1. If <u>a</u> is a root of f(x), then (x-a) must divide f(x) evenly; that is, there is no remainder when we perform the division:

   f(x) = (x-a)q(x)+r(x)

   Since deg[r(x)] < deg[x-a], deg[r(x)] = 0, and r(x) is some constant c. Letting x=a gives:

   f(a)=0 *(because a is a root)*

   f(a)=r(a)=c=0

   Thus, when dividing a polynomial by (x-a), where a is one of it's roots, there will be no remainder.
2. The remainder upon dividing f(x) by x-a is f(a):

   f(x) = (x-a)q(x)+r; Let x=a:

   f(a)=r (not necessarily 0 if a is not the polynomial's root)

3. Rational Root Theorem: All rational roots are of the form p/q where p and q are relatively prime integers, p divides a0 (the constant term) evenly, and q divides a[n] (the coefficient of the term with the highest degree) evenly.
   a. Example: Find roots of x^3-6x^2+11x-6

      We know that the roots must be factors of 6, by **rational root theorem**
4. Odd degree principle: If all of the degrees in a function are odd, then f(x)=-f(-x). This is called an odd function.
5. **Double roots** are roots that occur twice. Often times, we can prove that there must be a **double root (**when all but one of the roots are given) by using the fact that their can never be one irrational or imaginary root if all coefficients are rational.
6. **Descartes' Rule of Signs**: This rule states that the maximum number of positive roots is determined by the number of sign switches that occur viewing a polynomial from left to right.
   a. Example: f(x)=3x^5+2x^4-3x^2+2x-1

      Here there are 3 sign switches and thus, a maximum of three roots

      Note: *f(-x) can be used to determine the maximum number of negative roots*
7. **Transforming Polynomials:**
   a. Example: Find the polynomial whose roots are the recipricals of the roots of x^4-3x^2+x-9

      Let f(x) = x^4-3x^2+x-9;

      f(1/x) will have roots that are the reciprocal of f(x)

      Let f(1/x)=1/x^4-3/x^2+1/x-9

      Let g(x)=1-3x^2+x^3-9x^4=0

      g(x)=-9x^4+x^3-3x^2+1 [coefficients are <u>**reversed**</u>]
8. **Newton's sums:** family of equations which relates to the sum of the mth powers of the roots of a polynomial:

Let $s[m]$ be the sum of the mth power roots of $f(x)$

$a[n]*s[1]+a[n-1]=0$

$a[n]*s[2]+a[n-1]*s[1]+2*a[n-2]=0$

$a[n]*s[3]+a[n-1]*s[2]+a[n-2]*s[1]+3*a[n-3]=0$

…

a. Example: Find the sum of the cubes of the solutions of $x^2-3x+3=0$;

   $1*s[1]-3=0; \Rightarrow s[1]=3;$

   $s[2] * 1 - 3 * s[1] + 3*2-=0; \Rightarrow s[2] = 3$

   $s[3]*1-3*s[2]+3*s[1] =0; \Rightarrow s[3]=$**0**

<u>Problems:</u>

Problem 1: If $q1(x)$ and $r1$ are the quotient and remainder, respectively, when the polynomial x^8 is divided by $x+1/2$, and if $q2(x)$ and $r2$ are the quotient and remainder, respectively, when $q1(x)$ is divided by $x+\frac{1}{2}$, then find r2.

$f(x) = x^8 = q1(x)*(x+1/2)+r1$

$f(-\frac{1}{2})=(\frac{1}{2}^8)=r1$

Thus, $r1=(\frac{1}{2})^8$

$q1(x)*(x+\frac{1}{2})=x^8-(\frac{1}{2})^8 = (x^4+(\frac{1}{2})^4)(x^2+(\frac{1}{2})^2)(x+1/2)(x-1/2)$

$q1(x)=(x^4+(\frac{1}{2})^4)(x^2+(\frac{1}{2})^2)(x-1/2)$

$q1(x)=(x+1/2)h(x)+r2$

$q1(-\frac{1}{2})=r2$.

Thus, $r2=\frac{1}{8}*\frac{1}{2}*(-1) = -1/16$

Problem 2: Solve the equation $(x+1)(x+2)(x+3)(x+4)=-1$ (**Algebra Manipulation**)

$(x^2+5x+4)(x^2+5x+6)=-1$

Let $x^2+5x=a$ and solve;

This is a common strategy when dealing with groups of consecutive integers

Problem 3: Give the remainder when $x^{203} -1$ is divided by x^4-1

$f(x)=x^{203}-1=h(x)*(x^4-1)+r(x)$

When $x^4=1$, $r(x)=f(x)$;

Thus, $r(1)=0,r(-1)=-2,r(i)=-1-i,r(-i)=i-1$

Let $r(x)=ax^3+bx^2+cx+d$;

Now we can sub all the pairs and solve for the coefficients and constants.

Problem 4: Given the equation $\left(x^2-3x-2\right)^2-3\left(x^2-3x-2\right)-2-x=0$ prove that the roots of the equation x^2-4x -2 =0 are roots of the initial equation and find the other roots as well.

We can manipulate the second equation and form: x^2-3x-2=x;

We know that, if the roots of the second equation are roots of the initial equation, then they should "intersect" at two positions on the x-axis. Finding the intersection:

x^2-3x-2-x=0

x-x=0;

Thus, they intersect at y=0 at both positions.

We reduce the initial polynomial to: $x^4-6x^3+2x^2+20x+8=0$

Since x^2-4x-2 is a factor, we write: (x^2-4x-2)(x^2+bx+c)

-4+b=-6

b=-2

-2*c=8

c=-4;

Thus, (x^2-4x-2)(x^2-2x+4) is equal to the initial equation.

Thus, 2+/- sqrt(6), 1+/- sqrt(5) are the roots of the initial equation.

Problem 5: If a,b,c,d are the solutions of the equation x^4-mx-3=0, then find the polynomial with leading coefficient 3 whose roots are: (a+b+c)/d^2, (a+b+d)/c^2 …

We know that a+b+c+d=0

Thus, (a+b+c)/d^2 = -1/d; Thus, we are really finding the negative recipricals of the roots of the equation.

f(1/x) has reciprocal roots.

f(1/x)=-3x^4-mx^3+1

f(-1/x) has negative reciprocal roots.

f(-1/x)=-3x^4+mx^3+1

We can multiply the entire polynomial by -1, yeilding:

-f(-1/x)=**3x^4-mx^3-1**

Problem 6:

4. The polynomial equation $x^3 - 6x^2 + 5x - 1 = 0$ has three real roots a, b and c.

   (a) Determine the value of $a^5 + b^5 + c^5$.

   (b) If $a < b < c$, show that c^{2004} is closer to its nearest integer than c^{2003} is to its nearest integer.

(b) We need to prove that a^n+b^n+c^n is always an integer; and then estimate the values of a^n+b^n.

**Hasn't been completed

Problem 7:

Problem 3.9 (2020 AIME I). [7] Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

Problem 8:

The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

(A) 7 **(B)** 8 **(C)** 16 **(D)** 17 **(E)** 18

Almost always, whenever provided with an incomplete quadratic equation with a target of complete versions with integer roots, the solution lies by using the discrimint and factoring it effectively.

a^2-8a=n^2

Now, factoring it in the typical way is ineffective here. However, it should be easy to see that a^2-8a+16 is factoriable, so we add 16 to both sides.

(a-4)^2=n^2+16

(a+n-4)(a-n-4)=16

Now that the expression is factored, we can use simple number theory to proceed and solve.

<u>Quadratic/Cubic Equations:</u>

Quadratic Equation: $ax^2+bx+c=0$;
Cubic Equation: $ax^3 + bx^2 + cx + d = 0$;

Formulas:

$x1,x2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Delta = b^2-4ac; *If delta is positive the results are integer values. This means that factoring is possible. If it is 0 then it has one real number solution. If it is negative than neither solutions are real numbers;*

The sum of the roots = -b/a (any degree equation);
x1*x2=c/a; (sum of the products of all the combinations of two roots)
x1*x2*x3=-d/a; (sum of the products of all the combinations of three roots)
…
(Essentially, the product of all the roots to any equation is (the constant)/(a). The sign will be positive if the degree of the equation is odd and even if not. The sum of the roots will always be -b/a.)

<u>Problems involving the formulas above:</u>

Problem 1 (Algebraic Manipulation using the sum and product formulas): If r and s are the roots of the equation $ax^2 + bx + c = 0$, what is the value of $\frac{1}{r^2} + \frac{1}{s^2}$ in terms of a,b,and c?

Strategy: Reduce the expression given by formulas provided and general fraction manipulation knowledge.

We can first reduce $\frac{1}{r^2} + \frac{1}{s^2}$ => $\frac{(s+r)^2-2sr}{r^2s^2}$
By the formula r+s=b/a and x1*x2=c/a, we can reduce it further:
$\frac{(b/a)^2-2*(c/a)}{(c/a)^2}$ => $\frac{b^2-2ac}{c^2}$

Problem 2 (Basic Implementation of quadratic factoring):

27. ___________ mi/h The Millers went on a weekend outing 180 miles from their home. The average speed to their destination was 20 mi/h less than the average speed returning home. If the travel time for the entire trip was 7.5 hours, what was the Millers' average speed to their destination?

Strategy: Represent the information given with equations, and solve for speed;

$st1 = 180 \;=>\; t1 = 180/s$
$(s + 20)t2 = 180$
$t1 + t2 = 7.5 \;=>\; t2 = 7.5 - t1$
$(s + 20)t2 = 180 \;=>\; (s + 20)(7.5 - t1) = 180 \;=>\; 7.5s + 150 - st1 - 20t1 = 180 \;=>$
$7.5s^2 - 3600 - 210s = 0 \;=>\; s^2 - 480 - 28s = 0 \;=>\; (s - 40)(s + 12) \;=>\; s = 40$

Problem 3 (Number theory and basic quadratic factoring):

What is the sum of all real values of x that are solutions to the equation $\left(\frac{2}{3}x^2 - x - \frac{2}{3}\right)^{(x^2 - 9x + 20)} = 1$? Express your answer as a common fraction.

There are three cases in which the following expression equals 1:

1. $x^2 - 9x + 20 = 0$ and $\frac{2}{3}x^2 - x - 2/3 \,!= 0$

2. $\frac{2}{3}x^2 - x - \frac{2}{3} = 1$

3. $\frac{2}{3}x^2 - x - \frac{2}{3} = -1$ and $(x^2 - 9x + 20)\ mod\ 2 = 0$

Case 1: (x-5)(x-4)=0; x=5,4; and in both cases the base isn't equal to 0;
Case 2: (2x-5)(x+1)=0; x=5/2.-1;
Case 3: (2x-1)(x-1)=0; x=1/2,1; when x=1, the exponent is even. When x=½ the exponent is not even. Thus, the only solution is x=1;
1+5+4+(-1)+5/2=18/2+5/2=23/2;

Problem 4: (Absolute Values)

What is the product of all the roots of the equation

$$\sqrt{5|x|+8} = \sqrt{x^2-16}.$$

(A) -64 **(B)** -24 **(C)** -9 **(D)** 24 **(E)** 576

Strategy: Manipulate the equation in terms of the absolute value of x and discuss in cases (keep in mind the extraneous solutions induced by squaring the equation at the start)

Squaring both sides and manipulating slightly we can get: |x|=(x^2-24)/5

Case 1: x=(x^2-24)/5

We can manipulate this expression to get (x+3)(x-8)=0
x=-3,8

Case 2: x=(-x^2+24)/5

We can manipulate this expression to get (x-3)(x+8)
x=3.-8

To mitigate against the extraneous solutions, we need to substitute the solutions back into the original equation;
sqrt(5|-3|+8)=sqrt(x^2-16)
sqrt(23)=sqrt(-7) Thus, both -3 and 3 don't work.
sqrt(5*8+8)=sqrt(8^2-16)
sqrt(48)=sqrt(48) Thus, both 8 and -8 are valid solutions.
Ans = (A) || -64

Problem 5 (Basic implementation of formulas):

The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

We want the polynomial $x^3 - ax^2 + bx - 2010$ to have POSITIVE integer roots. That means we want to factor it in to the form $(x-a)(x-b)(x-c)$. We therefore want the prime factorization for 2010. The prime factorization of 2010 is $2 \cdot 3 \cdot 5 \cdot 67$. We want the smallest difference of the 3 roots since by Vieta's formulas, a is the sum of the 3 roots.

We proceed to factorize it in to $(x-5)(x-6)(x-67)$. Therefore, our answer is $5+6+67$ = $\boxed{\textbf{(A)} 78}$.

Problem 6 (Basic implementation of formulas): The solutions of x^2+bx+c=0 are each 5 more than the solutions of x^2+7x+3=0. What are the values of b and c? Express your answer as an ordered pair (b,c).

The sum of the solutions in the second equation is -b/a=-7

If both solutions to the first equation are five greater than those of the second then the sum of the solutions must be 5*2=10 greater than the sum of the second. Thus, the sum of the two roots of the first equation is 3. So, -b/a=3; a=1,b=-3; Let e,d = the roots of the second equation.

(e+5)(d+5)=ed+5(e+d)+25=3+5*-7+25=-7. (b,c)=(-3,-7)

Problem 7 (basic cubic formula implementation): A cubic equation of the form x^3+bx^2+cx+d=0 has solutions x=3, x=4 and x=5. What are the values of b,c, and d?

-b=3+4+5=12; b=-12

-d=60; d=-60

c=12+20+15=47

Problem 8 (basic cubic formula and algebraic manipulation): What is the sum of the reciprocals of the solutions of x^3-3x^2-13x+15=0;

We can assume the three roots of the cubic equation to be x1,x2,x3 such that 1/x1+1/x2+1/x3 is the sum of the reciprocals of its solutions.

If we multiply the fractions by the lcm of the denominators, x1x2x3, then we get the fraction: (x1x2+x2x3+x1x3)/(x1x2x3). Coincidentally, the numerator and denominator and both can be separately solved by using the equations. (c/a)/(-d/a)=(c/a)*(-a/d)=-c/d=13/15

Problem 9 (basic cubic formula and algebraic manipulation): What is the sum of the squares of the solutions of x^3-15x^2+66x-80=0?

Let x1,x2,x3 be the solutions to the cubic equation.

(x1+x2+x3)^2=x1^2+x2^2+x3^2+2(x1x2+x1x3+x2x3); => (-b/a)^2=ans+2*(c/a); => ans=225-132=93

Problem 10 (basic cubic formula + number theory): The solutions of x^3-63x^2+cx-1728=0 form a geometric sequence. What is the value of c?

Let z,zy,zy^2 = the roots of the cubic equation;

$z^3y^3=1728 \Rightarrow zy=12$;

$z+zy+zy^2=63$; $\Rightarrow$ $z+zy^2=63-12=51$; *$z(1+y^2)=51$; $1+y^2=17$; $\Rightarrow$ $y=4, z=3$; (I assumed that the roots had to be integer despite the question not mentioning so)*

$c=y^2(z+z^2+z^3)=9(4+16+64)=756$;

Problem 11 (basic cubic formula + number theory): The cubic equation $x^3-10x^2+Px-30=0$ has three positive integer roots, determine the value of P.

Let x_1,x_2,x_3 be the roots of the cubic equation.

$x_1+x_2+x_3=10$;

$x_1x_2x_3=30$; x_1,x_2,x_3 are factors 30. Thus, $x_1,x_2,x_3=2,5,3$.

$P=x_1x_2+x_1x_3+x_2x_3=10+15+6=31$;

What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)+2019$$

where x is a real number?

(A) 2017 **(B)** 2018 **(C)** 2019 **(D)** 2020 **(E)** 2021

Problem 12:

Essentially, the problem is asking for the minimum value of the expression $(x+1)(x+2)(x+3)(x+4)$.

We can combine the values on the opposite ends of the multiplication expression, producing $(x^2+5Ax+4)(x^2+5x+6)$

Let $x^2+5x=z$;

Thus, $f(z)=(z+4)(z+6) \Rightarrow f(z)=z^2+10z+24$;

Completing the square, $f(z)=(z+5)^2-1$; $f(z)$ has a minimum value of -1. Thus, the minimum possible value of the expression is 2018.

Problem 13:

Let a be the largest real value of x for which $x^3-8x^2-2x+3=0$. Determine the integer closest to a^2.

We can use an important property here: Let $f(x) = x^3-8x^2-2x+3$; If $f(n)>0$ and $f(m)<0$, then a root must be between m and n; Let the roots be x_1,x_2,x_3 such that $x_1<x_2<x_3$;

By testing values:

$8<x_3<9$

$0<x_2<1$

$-1<x_1<0$

These restrictions are unique, because x3 is significantly large than x2 and x1;
Testing a few more values we know that:

$1/2<x2<1/\text{sqrt}(2)$

$-1<x3<-½$

Thus, $1/2<x2^2+x3^2<3/2$

$x1^2+x2^2+x3^2=(x1+x2+x3)^2-2(x1x2+x1x3+x2x3) = 64-2*-2=68$

$x3^2=68-(½ => 3/2)$

Thus, $66.5<x3^2<67.5$

Thus, $x3^2=$**67**

Algebraic Manipulations:

Common Formulas:

$a^2 – b^2 = (a-b)(a+b)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 – 2ab + b^2$

$(a+b)^3 = a^3+ 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3- 3a^2b + 3ab^2 – b^3$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

x^k-y^k = (x-y)(x^[k-1]+x^[k-2]*y+... xy^[k-2]+y^[k-1])

$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc$ = (a-b)^2+(a-c)^2+(b-c)^2

Strategies:

1. Represent expressions with variables to solve for the expressions (xy=a, x+y=b, etc.)
2. Let's say we are given two of the quantities xy, x+y, x^2+y^2, x^3+y^3, etc., and asked for a third. We can solve for the value using the following patterns/tricks/formations:
    a. If we are given ab and a+b and the target is $a^2 + b^2$: $(a+b)^2 = a^2 + 2ab + b^2$ -> $\underline{(a+b)^2 - 2ab = a^2 + b^2}$
    b. If we are given ab and a+b and the target is a^3+b^3= (a+b)(a^2-ab+b^2) -> We find a^2+b^2 using the method in (a) and then substitute known values into the formula.
    c. Similar approaches can be used for other values
3. Represent the relationship between constants by assuming variables and factorizing:
    a. 50(51)(52)(53) = (x-2)(x-1)x(x+1) = (x^2-x-2)(x^2-x) [where x=52]
4. (x^4a+y^4b) = (x^2a+y^2b)^2-2x^2ay^2 = (x^2a+y^2b+2x^a*y^a)(x^2a+y^2b-2x^ay^b)
    a. Useful to find close factors + factoring forms
5. When given a system of equations adding, subtracting, dividing, or multiply all of the systems is oftentimes a valid and efficient strategy to be mindful of.
6. Partial Fractions -> When dealing with rational functions

Types of Problems:

1. Manipulating toward a common formula
2. Manipulating Quadratics (Using discrimint)
3. Manipulating Divisibility
4. Symmetry in equations

5. Telescoping + Partial Fraction

<u>Manipulating toward a formula:</u>

Problem 1: Find $x^2 + y^2$ if x and y are positive integers such that $xy + x + y = 71,$ $x^2y + xy^2 = 880.$

Strategy: Represent expressions with variables and combine the equations. Then, use the factor of the equation and solve for the positive values.
Let x+y=a and xy =b;
a+b=71;
$x^2y+xy^2=880$; => xy(x+y)=880; => ab=880;
880/b+b=71;
$b^2-71b+880=0$;
(b-55)(b-16)=0; => b=55,16; => a=16,55;
If b=55 and a=16 then it is easy to see that 5 and 11 is a plausible solution
If b=16 and a=55 then there is no solution because no two factors of 16 can have a sum greater than 17.
Thus, $5^2+11^2=25+121=146$

Problem 2:

20. ______________ For integers a, b, c and d, $(x^2 + ax + b)(x^2 + cx + d) = x^4 + x^3 - 2x^2 + 17x - 5$. What is the value of $a + b + c + d$?

If we combine the two expressions in the brackets and combine the like terms mentally, we can form some equations around the coefficients of each term.

1. a+c=1;A
2. b+d+ac=-2;
3. bc+da=17;
4. bd=-5;

We know that b,d are integers such that b and d must be factors of -5; (1,-5) || (5,-1);
Instead of iterating through all 4 possibilities for b,d and substituting their values, we can just substitute the sum of b and d into equation 2 and solve using equation 1. We have two cases for the sum of b,d; 4 or -4. If a+c=1, then there are two cases for the values of a and c: 1. a=0 || c=0; 2. a<0 || c<0;

It is impossible for a or c to be equal to 0 given the condition abcd!=0. Thus, ac must be negative and b+d must equal 4. Thus, a+b+c+d=5;

Problem 3: The nonzero roots of the equation x^2+ 6x+k = 0 are in the ratio 2:1. What is the value of k?

We know that a/b=2;
We can also represent this equation as (x-a)(x-b)=0; => x^2-(a+b)+ab;
Because this equation is equal to x^2+6x+k; We now know that a+b=-6;

We can deduce through the equation a/b=2, that a,b=-2,-4; k=ab; Thus, k=8;

Problem 4: Express 2^22+1 as the product of two four-digit numbers.

2^22+1 = (2^11+1)^2 - 2^12 = (2^11+1-2^6)(2^11+1+2^6)

Vieta's Theorem Manipulations:

Problem 1:

Problem

The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

$\textbf{(A)}\ 255 \qquad \textbf{(B)}\ 256 \qquad \textbf{(C)}\ 257 \qquad \textbf{(D)}\ 258 \qquad \textbf{(E)}\ 259$

Solution 1

The sum of the first m odd integers is given by m^2. The sum of the first n even integers is given by $n(n+1)$.

Thus, $m^2 = n^2 + n + 212$. Since we want to solve for n, rearrange as a quadratic equation: $n^2 + n + (212 - m^2) = 0$.

Use the quadratic formula: $n = \dfrac{-1 + \sqrt{1 - 4(212 - m^2)}}{2}$. Since n is clearly an integer, $1 - 4(212 - m^2) = 4m^2 - 847$ must be not only a perfect square, but also an odd perfect square for n to be an integer.

Let $x = \sqrt{4m^2 - 847}$; note that this means $n = \dfrac{-1 + x}{2}$. It can be rewritten as $x^2 = 4m^2 - 847$, so $4m^2 - x^2 = 847$. Factoring the left side by using the difference of squares, we get $(2m + x)(2m - x) = 847 = 7 \cdot 11^2$.

Our goal is to find possible values for x, then use the equation above to find n. The difference between the factors is $(2m + x) - (2m - x) = 2m + x - 2m + x = 2x$. We have three pairs of factors, $847 \cdot 1$, $121 \cdot 7$, and $77 \cdot 11$. The differences between these factors are 846, 114, and 66 - those are all possible values for $2x$. Thus the possibilities for x are 423, 57, and 33.

Now plug in these values into the equation $n = \dfrac{-1 + x}{2}$, so n can equal 211, 28, or 16, hence the answer is $\boxed{\textbf{(A)}\ 255}$.

Divisibility Manipulations:

Determine all integers a for which $\dfrac{a}{1011 - a}$ is an even integer.

The goal with problems like these that have a similar structure is to convert the fraction such that there is a constant in the numerator and a variable in the denominator

a/(1011-a)=(a-1011+1011)/(1011-a)=-1+(1011)/(1011-a)

Thus, 1011-a must be a factor of 1011.

1011 = 3*337; Thus,

1011-a = 1, 3, 337, 1011, -1, -3, -337, -1011

a = 1010, 1008, 674, 0, 1012, 1014, 1348, 2022

Self-Symmetry:

Problem 1: $\sqrt[3]{3-2\sqrt[3]{3-2\sqrt[3]{3-2\sqrt[3]{3-\ldots}}}}$

We let x = cbroot(3-2cbroot(3-2cbroot(3-...)))

Thus, x=cbroot(3-2x)

x^3=3-2x;

x^3+2x-3=0;

x=1 is the only real solution. Thus, this expression simplifies to 1;

Problem 2: $3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\ldots}}}$

Let x= 3+1/(2+1/(3+1/(2+...)))

Thus, x= 3+1/(2+1/x)

(x-3)(2+1/x)=1

2x-6+1-3/x=1

2x^2-6x+x-3=x

2x^2-6x-3=0;

x=(3+sqrt(15))/2

<u>Miscinaneous Manipulations:</u>

A rectangular box measures $a \times b \times c$, where $a, b,$ and c are integers and $1 \leq a \leq b \leq c$. The volume and surface area of the box are numerically equal. How many ordered triples (a, b, c) are possible?

(A) 4 **(B)** 10 **(C)** 12 **(D)** 21 **(E)** 26

2(ab+bc+ac) = abc

This is a common form in which the following algebraic manipulation can be carried out by dividing by abc:

½=1/c+1/a+1/b

This form of equation can be solved using a bounding strategy:

Let a<=b<c such that 1/a>=1/b>=1/c

3/a>=½>=3/c

c>=6>=a>=3

a=3: ⅙=1/c+1/b => 2/b>=1/6>=2/c => c>=12>=b>=6

b=7, c=42

b=8, c=24

b=9, c=18

b=10, c=15

b=12, c=12

5 pairs

a=4; ¼=1/b+1/c => 2/b>=1/4>=2/c => c>=8>=b>=5

b=5,c=20

b=6, c=12

b=8, c=8

3 pairs

a=5: 3/10=1/b+1/c; 2/b>=3/10>=2/c; c>=6, 6>=b>=5;

b=5, c= 10

1 pair

a=6: 1/b+1/c=⅓; 2/b>=⅓>=2/c; c>=6>=b

b=6, c=6

1 pair

10 pairs total 🙂

<u>Telecoping + Partial Fractions:</u>

Summary: Telescoping problems ask the solver to simplify an expression that contains a unique pattern that results in an interesting simplification that can be used to solve the problem quickly and efficiently.

Strategies:

1. Split to partial fractions and induce a cancellation
2. Recognize that the series looks similar to a simpler series and separate the terms to produce that simpler series
3. Use bounding to find the range of the functional sum

Problem 1: (Partial Fractions)

Let $f(x) = \dfrac{1}{x^3 + 3x^2 + 2x}$. Determine the smallest positive integer n such that

$$f(1) + f(2) + f(3) + \cdots + f(n) > \frac{503}{2014}.$$

This problem involves a method called partial fractions. I should think of this method when encountering a rational function.

f(x)=1/x(x+1)(x+2) = a/x+b/(x+1)+c/(x+2) = [a(x+1)(x+2)+b(x)(x+2)+c(x)(x+1)]/[x(x+1)(x+2)] =

$$\frac{(a+b+c)x^2+(3a+2b+c)x+2a}{x(x+1)(x+2)}.$$

a=½,b=-1, c=½;

$$f(x)=\frac{1}{2}\left(\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}\right).$$

From these reductions, using the partial fraction method, we can now see the pattern with f(1)+f(2)+...f(n) Specifically, it is equal to:

$$\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+2}\right).$$

From this, we get:

$$\frac{1}{4}-\frac{1}{2(n+1)(n+2)}>\frac{503}{2014}.$$

Now, we may reduce this to: n=44;

Exercise 1.27. Sum the series $\sum_{j=1}^{\infty}\frac{2j-1}{2^j}=\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\cdots.$

Problem 2:

Here we notice that the series is suspiciously similar to ½+¼+⅛+1/16+...

$$\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\cdots.$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$$
$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$$
$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$$
$$\frac{1}{8}+\frac{1}{16}+$$
$$\frac{1}{8}+\frac{1}{16}+$$
$$\frac{1}{16}+$$
$$\frac{1}{16}+$$

Adding up the horizontal rows:

1. ½+¼+... = (½)/(1-1/2)=1
2. ¼+⅛+... = ½
3. =½
4. ⅛+1/16+...=(1/8)/(1/2)=¼
5. ¼
6. ⅛
7. ⅛
8. 1/16
9. 1/16

…

Now we can lay out the pattern as follows: 1,1,½,¼,⅛+...

Thus, the sum is 1+(1+½+¼…)

1+(1)/(1-1/2)=3

Problem 3: $$\text{Find} \sum_{i=1}^{\infty} \frac{1}{i \cdot (i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$$

Here, we may invoke partial fractions:

1/(i(i+1))=a/i+b/(i+1);

a(i+1)+bi=1

ai+bi+a=1 (thus, a=1)

i+bi=0

i(b+1)=0 (thus, b=-1)

Thus, $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

The sequence now becomes: 1/1-½+½-⅓+1/4+-¼… Thus, the sequence will ultimately = 1;

Problem 4:

Example 1.29 (2018 Math Prize for Girls). Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}}.$$

Determine $\lfloor S_{4901} \rfloor$. Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

This problem involves a bounding approach that is hard to come by:

The motivation is as follows: Because the problem involves upper and lower bounds, it is a could idea to bound the actual result with two easier-to-find results whose products/sums telescope.

A sum in a form similar to sqrt(n+2)-sqrt(n) is convenient;

We know S1=1; Thus, let Tn+1=Sn;

$$\sqrt{2k-1} + \sqrt{2(k-1)-1} < 2\sqrt{2k-1} < \sqrt{2(k+1)-1} + \sqrt{2k-1}.$$

We can extrapolate the condition above because every subsequent term is larger than the last. Now, our goal becomes to convert the inner fraction to the summation provided:

$$\frac{\sqrt{2k-1} - \sqrt{2(k-1)-1}}{2} > \frac{1}{2\sqrt{2k-1}} > \frac{\sqrt{2(k+1)-1} - \sqrt{2k-1}}{2}.$$

=> $$\sum_{k=2}^{n} \left(\sqrt{2(k+1)-1} - \sqrt{2k-1}\right) < T_n < \sum_{k=2}^{n} \left(\sqrt{2k-1} - \sqrt{2(k-1)-1}\right)$$

When n=4901, $\sqrt{9803}-\sqrt{3}<T_{4901}<\sqrt{9801}-\sqrt{1}$ (This cancels because we purposely produced the terms in the form: sqrt(n+2)-sqrt(n))

We approximate the square roots and find that: Thus, $\lfloor T_{4901}\rfloor = 97$, and so $\lfloor S_{4901}\rfloor = 98$.

Problem 5:

Consider the sequence defined by $a_k = \dfrac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = 1/29$, for positive integers m and n with $m < n$, find $m+n$.

1/n(n+1)=1/n-1/(n+1); Thus, the equation give becomes:

1/m-1/n=1/29;

Thus, we can solve for possible values of m and n through simons factoring trick.

Binomial Theorem:

What is the hundreds digit of 2011^{2011}?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 9$

Problem 1:

It's always a good idea to question "why were these numbers chosen?". In this case, the number 11 was chosen because of its relationship with the number 10.

Notably, 2011^n (mod 1000) is equal to 11^n (mod 1000)

Now, we can represent 2011 = 11 = (10+1)^2011

By binomial theorem we know (10+1)^2011 = 1^2011 + 2011C1 * 10 + 2011C2 * 10^2 (mod 1000) which can easily be calculated and solved :).

Inequalities:

Important Theroems and Strategies:

1. Be careful of negative multiplication and division rules**
2. The **Trivial Inequality** states that the square of any real number is nonnegative. In other words, if x is real, then x^2>=0. Equality only holds when x=0.
3. When finding the maximum or minimum of a quadratic expression, complete the square and use the Trivial Inequality.
4. To combine inequalities we employ a similar strategy to when we combine ratios.

5. **AM-GM**: States that $(a+b)/2>=\text{sqrt}(ab)$ and can be generalized for n numbers
    a. We can use AM-GM to prove inequalities
    b. We can also use AM-GM to find maximum values for things (which we will do in a few questions below)
        i. The equality of AM-GM is satisfied when all numbers are equal
6. QM-AM-GM-HM is the generalized version of AM-GM that takes into account the Harmonic and Root Mean Formulas.

Problem 1: (Combining Inequalities + Factorization Patterns + Discrimint)

Question C1 (10 points)

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NO PHOTOCOPIES!

(a) Find all integer values of a such that equation $x^2 + ax + 1 = 0$ does not have real solutions in x.

(b) Find all pairs of integers (a, b) such that both equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

have no real solutions in x.

(c) How many ordered pairs (a, b) of positive integers satisfying $a \leq 8$ and $b \leq 8$ are there, such that each of the equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

has two unique real solutions in x?

(a) $a^2-4<0$; Thus, $a^2<4$; Thus, $a<2$ and $a>-2$
(b) $a^2-4b<0$ and $b^2-4a<0$

$a^2b<4b^2$ and $4b^2<16a$ [*strategy => combine inequalities*]
Thus, $ab<16$. One of a or b must be less than 4. Thus, if $a<4$. Then $b^2<16$, thus, $b<4$. The same applies the other wa around. Thus, $a,b<4$. By testing all the possible values, we find that $a,b=(1,1),(2,2),(3,3)$.

(c) $a^2-4b>0$, $b^2-4a>0$; By the a similar factorization from (b), $ab>16$. At least one of a or b is greater than 4. Let $a>4$. Thus, $b^2>16$. Thus, $a,b>4$. Then, by counting we get **12** solutions total.

Problem 2: $x^3-12x^2+ax-64$ has real, nonegative roots, find a.

$x_1+x_2+x_3=12$,
$x_1x_2x_3=64$

By AM-GM:

$(x_1+x_2+x_3)/3 >= \text{cuberoot}(x_1x_2x_3)$
Thus, $12/3>=\text{cuberoot}(64)$
$4>=4$;

The only case in which the AM-GM inequality is an equality occurs when all terms are equal.

Thus, x1=x2=x3=4;
Thus, a=48;

Problem 3: Find the smallest integer n such that (x^2+y^2+z^2)^2<=n(x^4+y^4+z^4)

x^4+y^4+z^4+2(x^2y^2+x^2z^2+y^2z^2) <= n(x^4+y^4+z^4)
2(x^2y^2+x^2z^2+y^2z^2) <= (n-1)(x^4+y^4+z^4)

By AM-GM: x^2y^2<=(x^4+y^4)/2
Thus, 2(x^2y^2+x^2z^2+y^2z^2) <= 2(x^4+y^4+z^4)

Thus, the minimum value for n is 3.

<u>Recurrence Patterns:</u>

Theorems and Strategies:

1. Whenever you are given a recurrence pattern:
    a. For example, $a_{n+1} = c_n a_n + c_{n-1} a_{n-1} + \cdots + c_{n-k} a_{n-k},$
        i. You may consider it's characteristic polynomial x^(n+1)+cn*x^(n)+...c_n-k*x^(n-k)
            1. This polynomial has roots r1,r2…
                a. The solution to the recurrence is a_n= c1*r1^n+c2*r2^n (where c1 and c2 represent random values that you must solve for given initial condition)

<u>Absolute Value and Roots:</u>

Problem 1:

(c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

We let a=x^2-4x-c;
Thus,
a-sqrt(8a)=0
a^2-8a=0
a=0,8;

Thus,
x^2-4x-c=0,8;

x^2-4x-c=0 has 2 roots when 16+4c>=0; or when c>=-4

x^2-4x-c=8 has 2 roots when 48+4c>=0; or when c>=-12

If c>=-4, the equation actually has 4 roots;

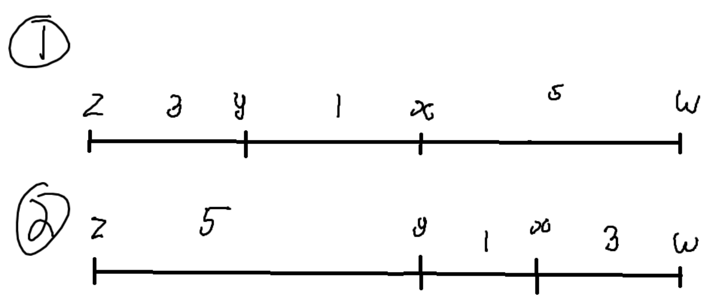
When c<=-12, the equation has 0 roots

When -4>=c>=-12, the equation has 2 roots. Thus, the range **-4>=c>=-12** is the solution.

Oftentimes, number lines can be implemented in both absolute value problems as well as problems involving positive pairwise differences:

Problem 2:

> Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are $1, 3, 4, 5, 6,$ and 9. What is the sum of the possible values for w?



Now that we have a clear diagram and two obvious cases, we can easily solve for the possible values of W.

Functions:

Strategies:

1. Oftentimes, it's a good idea to assume a value within a function, to prove some of it's properties:
    a. This can be useful when trying to determine the remainder
    b. This can also be useful when proving the range of a function
2. When two functions are equal for all real numbers x, then they are the same function with the same coefficients and constants
3. Contradiction method can be used when given an equation and a function that you must prove isn't equal to a value;
4. When given "floor()" functions, it can often be a good idea to abuse the fact that a floor function always outputs an integer (eg. by assuming the output as n [where n belongs to the set of all integers])

Problem 1:

4. For each positive integer n, define $f(n)$ to be the smallest positive integer s for which $1+2+3+\cdots+(s-1)+s$ is divisible by n. For example, $f(5)=4$ because $1+2+3+4$ is divisible by 5 and none of 1, $1+2$, or $1+2+3$ is divisible by 5.

   (a) Determine all positive integers a for which $f(a)=8$.

   (b) Prove that there are infinitely many odd positive integers b for which

$$f(b+1)-f(b)>2009\ .$$

(b)

$(b+1) | (n+1)n*\frac{1}{2}$

We know that n+1 and n must be of different parity such that it is convenient to let $b=2^n-1$

$(2^n) | (n+1)n*\frac{1}{2}$;

If $2|n+1$ then $(n+1)*\frac{1}{2} = 2^n$;

Thus, $n=2^{(n+1)}-1$

If $2|n$ then $n*\frac{1}{2} = 2^n$

Thus, $n=2^{(n+1)}$

We take the lower of these two cases so **$n=2^{(n+1)}-1$**

Thus, $f(b+1) = 2^{(n+1)}-1$

$(b) | (n+1)n*\frac{1}{2}$

$(2^n-1) | (n+1)n*\frac{1}{2}$

If n+1 is odd, then $n \le 2^n-2$

If n is odd, then $n \le 2^n-1$

We take the lower of these two cases so $n \le$ **2^n-2**

Thus, **$f(b) \le 2^n -2$**

The reason we use <= here is because is because even numbers can have odd numbers as their factors; thus, 2^n-1 could be distributed between (n+1) and n so there may be lesser values of f(b)

The final equation becomes:

$2^{(n+1)}-1 - 2^n-2 > 2009$ (for infinitely many cases)

This is obviously true, buecause any value where $n \ge 12$ satisfies this equation.

4. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.
For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$.
For $x > 0$, define $f(x) = \left(x + \frac{1}{x}\right) - \left\lfloor x + \frac{1}{x} \right\rfloor$.
For example, $f\left(\frac{4}{9}\right) = \left(\frac{4}{9} + \frac{9}{4}\right) - \left\lfloor \frac{4}{9} + \frac{9}{4} \right\rfloor = \frac{97}{36} - 2 = \frac{25}{36}$.
   (a) Determine all $x > 0$ so that $f(x) = x$.
   (b) Suppose that $x = \dfrac{a}{a+1}$ for some positive integer $a > 1$.
   Prove that $x \neq f(x)$, but that $f(x) = f(f(x))$.
   (c) Prove that there are infinitely many rational numbers u so that
   - $0 < u < 1$,
   - u, $f(u)$ and $f(f(u))$ are all distinct, and
   - $f(f(u)) = f(f(f(u)))$.

Problem 2:

(a) We first start by substituting and looking for patterns/facts:

1/x = floor(x+1/x)

We know that any output of the floor() function must be an integer. Thus, we can assume n=1/x (where n is an integer strictly greater than 1)

Thus, n=floor(n+1/n); (We know that n is an integer >1 and that 1/n<1)

Thus, n must equal n;

Thus, in order for the function f(x)=x to be satisfied, x=1/n (where n is a positive integer > 1)

(b) Here, we must use our conclusion from (a)

We know that for x!=f(x), x!=1/n (for an integer n>1)

For a/(a+1)=1/n; a must equal 1; However, a>1; So this condition can never be satisfied

For f(x)=f(f(x)); f(x)=1/n (for an integer n)

a/(a+1)+(a+1)/a-floor[a/(a+1)+(a+1)]=1/n

(2a^2+2a-1)/(a^2+a)-floor[2+1/(a^2+a)]=1/(a^2+a) which, for a>1, always equals 1/n, integer n. Thus, f(x)=f(f(x));

(c) The first steps toward solving this is to express all of the conditions through algebra:

We let f(u)=a/(a+1), f(f(u))=f(f(f(u)))=a/(a^2+a). (where u!=a/a+1 or a/(a^2+a))

We also know u is rationale and 0<u<1 thus, u=b/b+c;

f(u)=**c^2/(b^2+bc)** = a/(a+1)

Thus, c^2+1=b^2+bc; => 4b^2+4bc+c^2-5c^2=4; (2b+c)^2-5c^2=4;

Let 2b+c = d; d^2-5c^2=4 (This is pell's equation and is thus, satisfied for all positive integers if there is one integer pair) => d=7, c=3 satisfies the equation: thus, there are infinite integer resolute to this equation

Therefore, there exists an infinite family of rational numbers $u = \frac{b}{b+c}$ with the required properties.

Problem 3:

PART C: Question #4 (10 marks)

Let $f(x) = x^2 - ax + b$, where a and b are positive integers.

(a) Suppose $a = 2$ and $b = 2$. Determine the set of real roots of $f(x) - x$, and the set of real roots of $f(f(x)) - x$.

(b) Determine the number of pairs of positive integers (a, b) with $1 \leq a, b \leq 2011$ for which every root of $f(f(x)) - x$ is an integer.

SOLUTION SOON

Magic Shapes

Magic Shapes:

Strategies:

1. If you are given a specific set of numbers, then you can determine the total sum. (If you are using numbers 1-7, then the sum is 28)
2. Magic Squares: The sum of each row = The total sum / n (where n=the length of the square's sides)
3. Magic Triangles: The corners are repeated twice; Thus, the sum = the sum of the used numbers + the sum of the corners. (If a range is given, then expressing the sum of each side as S and the total as 3S may be helpful)

Types of Problems:

1. Magic Squares
2. Magic Triangles

Magic Squares:

Problem 1:

In a magic square, each row, each column and both main diagonals have the same total. In the partially completed magic square shown, what number should replace x?

18			
13	15		
	10	11	17
	x	16	14

Strategy: Recursively determine the values of boxes, eventually determining the target value.

Diagonal sum = 18+15+11+14=58
10+11+17+x=58; => x= 20;
18+13+20+x=58; => x=7;
16+14+7+x=58; => x=21

Problem 2:

In each row of the table, the third cell contains the sum of the contents of the first two cells.

In each column of the table, the third cell contains the sum of the contents of the first two cells.

What total would you get if you added up the contents of all nine cells?

△	4	△+4
8	☆	☆+8
△+8	☆+4	16

Strategy: Identify the relationship between the equation x+8+y+4=16 and the total sum;

x+8+y+4=16; => x+y=4;
Total: 3x+3y+8*3+4*3+16; => 3x+3y+52; 3(x+y)+52; => 3*4+52=**64**

Problem 3:

25. In the table, the numbers in each row form an arithmetic sequence when read from left to right. Similarly, the numbers in each column form an arithmetic sequence when read from top to bottom. What is the sum of the digits of the value of x?

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

(A) 5 **(B)** 2 **(C)** 10
(D) 7 **(E)** 13

				18
	43			
		40		
x			26	

Strategy: When no row or column has more than 1 given number, then we must use algebra to determine the target cell. *(A strategy with arithmetic sequences is to represent the difference in terms of a variable. An effective way to do this is to represent the cell connecting two already known cells)*

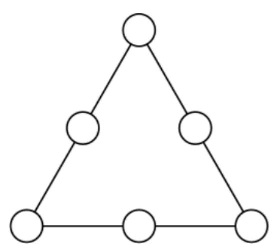
After using this strategy:

		40-2d		18
	43	40-d	37-2d	34-3d
		40		50-6d
		40+d		66-9d
x		40+2d	26	82-12d

We can clearly make an equation in terms d and 26 on the bottom row:

82-12d-40-2d+80+4d=52; => d=7;
Now, we can substitute this into the cell in row 5 column 3, 40+2d = 40+14=54;
We can see that the common difference on the bottom row is 54-26=28;
Thus, x=54+28*2=110;
Digit sum=2;
Magic Triangles:

In a magic triangle, each of the six whole numbers $10-15$ is placed in one of the circles so that the sum, S, of the three numbers on each side of the triangle is the same. The largest possible value for S is



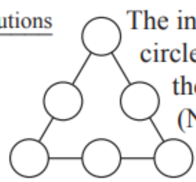
Problem 3: (A) 36 (B) 37 (C) 38 (D) 39 (E) 40

Strategy: Use the total sum of the sides to determine the value of S.

3s-(a+b+c)=75; => s-(a+b+c)/3=25; => To maximize (a+b+c), 13+14+15=42 is the max; s=25+42/3=**39**;

Problem 4:

13. ________ solutions



The integers 1 through 6 are to be used, each exactly once to fill the six circles in the figure so that the sums of three integers along each side of the triangle are the same. How many different solutions are possible? (Note that two solutions are considered the same if one can be rotated or reflected to obtain the other.)

Let a,b,c equal corners, and the remaining 3 variables equal the sides.
Let S = the sum of one side.
2(a+b+c)+d+e+f = 3S

3S-(a+b+c)=21.
S=7+(a+b+c)/3;
Thus, a+b+c must be divisible by 3.
The least value is 1+2+3=6; The greatest value is 4+5+6=15;
When their sum is 6: (1,2,3)
When their sum is 9: (2,3,4), (1,2,6), (1,3,5)
We do not need to consider if their sum is 12 or 15 because these cases are already counted.
So there are 4 possible solutions.

Problem 5:

The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 42 \qquad \textbf{(E)}\ 60$

Here, we can use two fundamental strategies:

1. Abuse the extremes (we know that 1 must be top left and 9 must be bottom right)
2. Case work with center (sometimes with different portion, but in these specific problems center is best)

Case 1: Center = 4

Case 2: Center = 5

Case 3: Center = 6

Set Theory

Formulas:

2 sets:
(A ∪ B) = (A) + (B) - (A∩B)
(AUB) = Universal Set - AUB;
3 sets:
(AUBUC)=(A)+(B)+(C)-(AUB)-(AUC)-(BUC)+(A∩B∩C);

Notation:

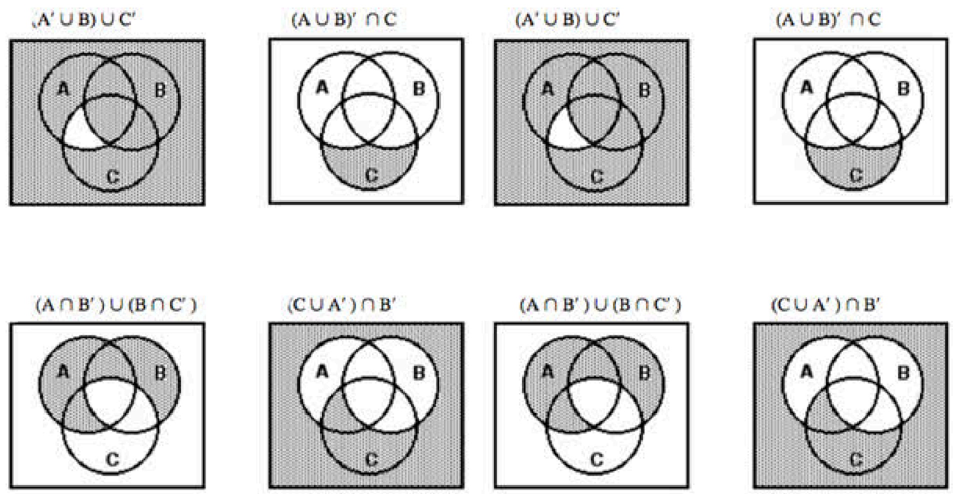
|A| = the number of elements inside A (oftentimes referred to as A for simplicity)
A’ or Ā = everything not included in A
A\B = everything included in A but not B.

Equivalent notations:

A∩(BUC) = (A∩B)U(A∩C)
AU(B∩C) = (AUB)∩(AUC)
(AUB)’=A’∩B’

Notice how the notation within the parenthesis acts similar to addition and the notation outside acts similar to *multiplication in polynomial distribution.*

Complex Notation Examples:



Types of Problems:

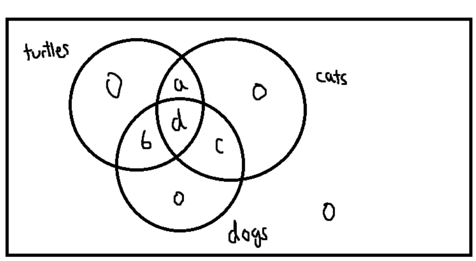
1. 2 sets
2. 3 sets
3. Multiples of numbers (either 3 or 2 sets)
4. Range sets

3 sets:

Problem 1: A town has 2017 houses. Of these 2017 houses, 1820 have a dog, 1651 have a cat, and 1182 have a turtle. If x is the largest possible number of houses that have a dog, a cat, and a turtle, and y is the smallest possible number of houses that have a dog, a cat, and a turtle, then x − y is what?

(1) To maximize the intersection area of these three sets, the larger set (that of dogs) must contain the medium-sized set (that of cats) and the medium-sized set must contain the smallest set (that of turtles). Thus, the intersection will simply be the value of the smallest set, 1182.

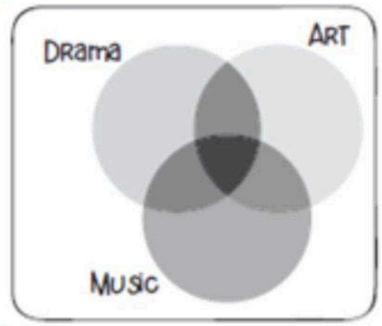
(2) To minimize the intersection area of these three sets, the number of intersections between 2 sets must be maximized. To do this, we can set the value outside the set as well as the value of every exclusively single set to 0.



The total sum of the intersections (a,b,c,d) will be 2017. Thus, a=2017-b-d-c=2017-1820=197. Thus, b=2017-1651=366. Thus, c=2017-1182=835. Thus, d=2017-1398=619.

Problem 2:

At Mesa Performing Arts Center, 30 students take courses in one or more of the drama, music and art departments. Five students take courses in exactly one department. Of these students, twice as many take drama courses as take music courses. Five students take courses in exactly two departments. Of these students, twice as many take drama and music courses as take music and art courses. Use the provided Venn diagram to organize this information, and then answer questions 11 through 13.



11.______ students How many students take only art courses?

12.______ students How many students take courses in all three departments?

13.______ students How many students take courses in art or music but not both?

A

11. If the combined number of students who took exactly one course is 5 and the number of students who chose only drama is twice that of only music then the only possible integer values for drama and music are 2 and 1 respectively. (given that 4 and 2 would be greater than 5). The number of students who chose only art will, thus, be 2.

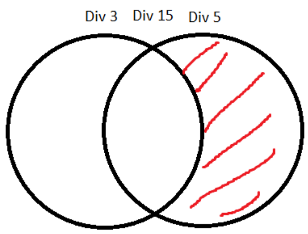
12. If the number of students who chose only one course is 5 and the number of students who chose only two courses is 5 as well, then the number of students who chose all three courses must be 30-(5+5)=20.

13. Following the logic from question 11, the number of students who did drama and music is twice of those who took music and art courses. Thus, the number of students who did drama and music is 2, the number of students who took music and art is 1, and the number of students who did drama and art is 2. Hence, 2+2+1+2=7 students did art or music but not both.

<u>Multiples Problems:</u>

Problem 1: (2 sets, multiples) How many two-digit positive integers are divisible by 5 but not by 3?

Strategy: Draw a diagram to better understand the target. Then, then solve for the number of 2-digit multiples of 3, 5 and 15. After, Subtract the number of 2-digit multiples of 5 from the multiples of 15 to achieve the desired result.



Divisible by 5 = floor($\frac{99}{5}$)=19

Divisible by 15 = floor($\frac{99}{15}$)=6

Subtracting the 1 digit multiples,
Divisible by 5 = 19 - 1
Divisible by 15 = 6 - 0
Target = 12

Problem 2: (Type: 3 set, multiples): Mrs. Sanders has three grandchildren, we call her regularly. One calls her every three days. One calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days of dying the next year did she not receive a phone call from her grandchildren.

Target: AUB
Strategy: Plug in known values into the formula and solve for AUB. Once this value is found, just subtract it from 365 to achieve the desired result;
Formula: AUBUC=A+B+C-AUB-AUC-BUC+A∩B∩C;

AUBUC=A+B+C-AUB-AUC-BUC+A∩B∩C; =
floor(365/3)+floor(365/4)+floor(365/5)-floor(365/12)-floor(365/15)-floor(365/20)+floor(365/60) = 121 + 91 + 73 -30 - 24 - 18 + 6 = 219;
Answer = 365-219=146 days;

Problem 1: Residents were surveyed in order to determine which flowers to plant in the new Public Garden. A total of N people participated in the survey. Exactly 9/14 of those surveyed said that the color of the flower was important. Exactly 7/12 of those surveyed said that the smell of the flower was important. In total, 753 people said that both the color and smell were important. How many possible values are there for N?

The premise of this question is that the people who chose to participate in the survey could have opted to choose neither option. With this, two clear extremes can be seen;

9/14N

Add pascal question (2017 ~ I think [maybe 18 or 19])

Geometry:

Plane Geometry:

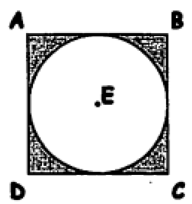
Solving for area:

There are five common ways to solve for the area of a shape:

1. Difference Method
2. Splitting (and moving) the shape
3. Using similarity and congruence laws
4. Using ratio between bases and heights (Applies to triangles)
5. Heron's Formula (Applies to triangle)

<u>Difference Method</u>

Problem 1 (Difference Method): Circle E is inscribed in square ABCD. If the length of segment AB is 4 inches, how many square inches are in the area of the shaded region? Express your answer in terms of pi .

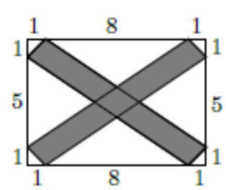


Target: Area of shaded region
Strategy: Subtract total area from the non-shaded area;

TotalArea = 4^2=16 in^2;
Area of Circle = 2^2 π
Shaded Area = 16 - 4 π;

Problem 2 (Difference Method + Similar Triangles):

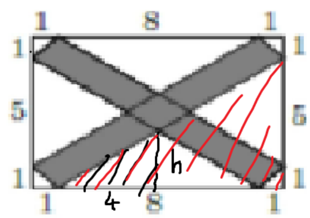
The rectangle below has length 10 cm and width 7 cm. An X-shaped figure (shaded) is drawn, with dimensions as shown. What is the area of the shaded figure, in cm^2? Express the answer as a common fraction.



General Strategy: Solve for the areas of the white triangles and subtract the total from the entire rectangle.

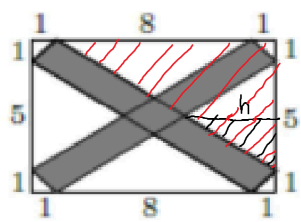
First, we can solve for the area of the smallest triangles. 1*½=½
There are 4 of these triangles: 4*½ = 2

Next, we solve for the triangles with a base of 8. The only issue is that we don't know the height. To determine the height, we can draw it as follow:



The triangle highlighted in black is similar to the triangle highlighted in red. This means that h/4=6/9=⅔
h=8/3 and the area of the triangle is 32/3. There are two of these triangles so their combined area is 64/3.

Next, we solve for the triangle with a base of 5. The only issue is that we don't the height. To determine the height, we can draw it as follows:

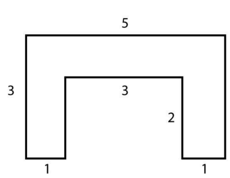


The triangle highlighted in black is similar to the triangle highlighted in red. This means that h/2.5=9/6=3/2. Thus, h=15/4. There are two of these triangles so their combined area is 75/4.

The total area is 70 and the total area of the white triangles is 2+64/3+75/2
The shaded area = 70-(2+64/3+75/2) = 335/12

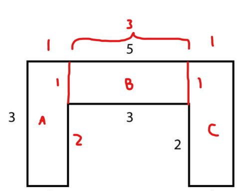
<u>Splitting Method:</u>

Problem 1:



Solve for the area of the figure above.

Target: Area;
Strategy: Split the complex shape into smaller calculable shapes and sum them up.

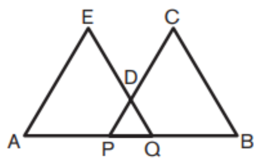


A = l*w=1*3=3; B=l*w=3*1=3; C=l*w=3*1=3;

Total Area=A+B+C=3+3+3=9;

Using Similarity:

Problem 1: In this figure, AE = EQ = BC = CP = 10 units, and AQ = BP = 12 units. The points A, P, Q and B are collinear. If the perimeter of the concave pentagon ABCDE is 52 units, what is its area? Express your answer as a common fraction.



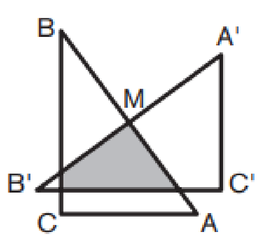
Target: Total Area => AEQ+PCB-PDQ;
Strategy: Use the similarity between the triangles to determine the side lengths of PDQ and then the area of PDQ. After solve for the total area by adding both large triangles and subtracting PDQ;

Triangle DPQ is similar to congruent triangles EAQ and CPB given that all sides of triangle DPQ are collinear to sides of the two larger triangles.

The ratio between the sides of the larger triangle is 10:10:12 = 5:5:6; We can construct an equation using this ratio;
(10+10+12)2-5x-5x-6x=52 => 12=16x; => x= ¾;
the Total area= Area of AEQ + Area of PCB - Area of PDQ;
AEQ area & PCB area = 48;

Ratio between large triangle side length : small triangle side length = 10: 15/4=8:3;
Ratio between areas = 64:9;
PDQ area = 9/64*48=27/4; Total Area = 96-27/4=384/4-27/4=357/4;

Problem 2: Right triangle ABC with AC = 3 units, BC = 4 units, and AB = 5 units is rotated 90° counterclockwise about M, the midpoint of side AB, to create a new right triangle A'B'C'. What is the area of the shaded region where triangles ABC and A'B'C' overlap? Express your answer as a common fraction.



Let the intersection of B’M and BC = D; the intersection of AM and B’C’=E; the intersection of B’C’ and BC = G;

Target: Area of the shaded region;
Strategy: Use similar triangle proportions to solve for the length of B’D. Then, use similar triangles to solve for the areas of B’ME and B’GD; After subtracting the areas to achieve the area of the shaded region.

Triangle BMD is similar to triangle BCA because they share an angle and both contain a 90-degree angle (AA proof). BM = 5/2; BC=4; Such that the ratio between the sides is 5/2:4=5:8; DM=⅝*3=15/8; B’M=20/8; Thus, B’D

= ⅝; Triangle BMD is congruent to triangle B’EM; ABC is also congruent to A’B’C’;

Thus, the ratio between B’M and B’C’ = ⅝; The ratio between the area of B’ME and B’C’A’ = 25/64; The area of B’ME is equal to 25/64*6=75/32; The ratio of B’D: B’A’ = ⅝:5=5:40=1:8; The ratio between the areas is 1:64; The area of B’DG is therefore 1/64 * 6=3/32;

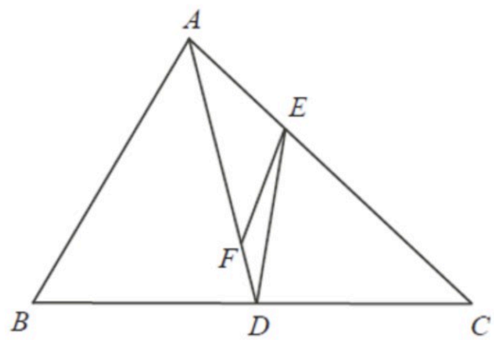
The area of the shaded region is, therefore, equal to 75/32-3/32=72/32=9/4;

<u>Using ratio between heights and bases (triangles)</u>

Property: If the ratio between the heights of two triangles are equal, then the area will be dictated by the ratio between their bases and vice versa.

Problem 1:

(b) In $\triangle ABC$, point D is the midpoint of side BC. Point E is on AC such that $AE : EC = 1 : 2$. Point F is on AD such that $AF : FD = 3 : 1$. If the area of $\triangle DEF$ is 17, determine the area of $\triangle ABC$.

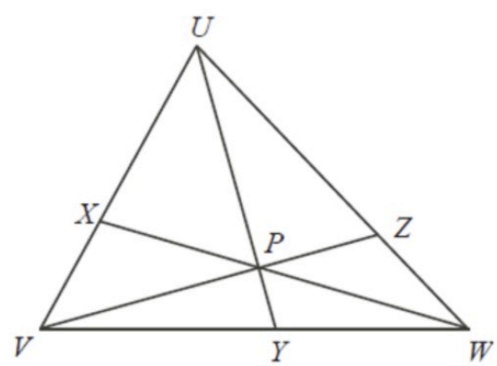


Strategy: Use the mentioned property to recursively find the ratio between each shape, eventually finding ABC.

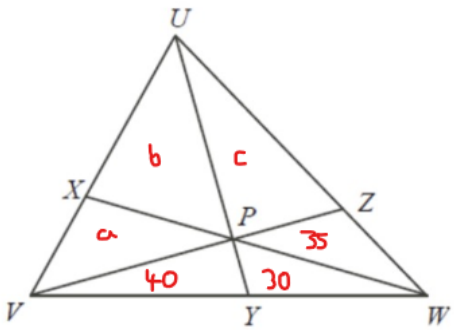
FDE : ADE = 1:4 because FD : AD = 1:4 and they share an equal height
ADE: ADC=1:3 because AE :AC=1:3 and the triangles share an equal height
ADC: ABD=1:1 because BC :DC=1:1 and they share an equal height
Thus, ABC: FDE=1:24;
ABC = 17 * 24 = 408;

Problem 2:

(c) In the diagram, points X, Y and Z are on the sides of $\triangle UVW$, as shown. Line segments UY, VZ and WX intersect at P. Point Y is on VW such that $VY : YW = 4 : 3$. If $\triangle PYW$ has an area of 30 and $\triangle PZW$ has an area of 35, determine the area of $\triangle UXP$.



Strategy: Use the ratio property mentioned to determine the area of triangle VPY and assume the values of the remaining shapes to be a,b, c respectively. After, create three equations (one for each side of the triangle) and solve for b;

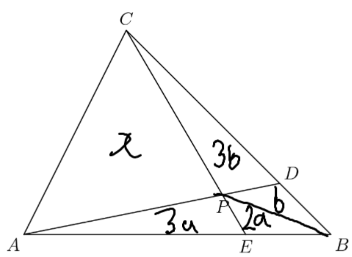


1. a+b/c+35=4/3;
2. c+35/30+40=b/a;
3. a+b/40+30=c/35;

a=56,**b=84**,c=70;

Problem 3:

In $\triangle ABC$ lines CE and AD are drawn so that $\frac{CD}{DB}=\frac{3}{1}$ and $\frac{AE}{EB}=\frac{3}{2}$. Let $r=\frac{CP}{PE}$ where P is the intersection point of CE and AD. Then r equals:



(A) 3 **(B)** $\frac{3}{2}$ **(C)** 4 **(D)** 5 **(E)** $\frac{5}{2}$

(x+3a)/(4b+2a)=3/2
(3b+x)/(5a+b)=3

x=6b
6b=15a

15a:3a = 5

<u>Heron's Formula:</u>

Formula: Let p=perimeter/2; A=√P(P-a)(P-b)(P-c));

Problem 1: Find the length of each side of an equilateral triangle having an area of 9 * √3).

Target: Length of sides;
Strategy: Assume values of sides and substitute into heron's formula;
Heron's Formula: A=√P(P-a)(P-b)(P-c));

let side =s; P=3s/2;
9 * √3) = √3s/2*(3s/2-s)^3) => (3s/2)*(s^3/8)= √3s^4/16)=√3)/4
* s^2=9*√3) = > s^2/36=1; => s=6;

Problem 2: The lengths of two adjacent sides of a parallelogram are 17 cm and
12 cm. One of its diagonals is 25 cm long. Find the area of the
parallelogram.

Target: Area of the parallelogram || area of singular triangle * 2;
Strategy: Substitute values into formula and solve;
Formula: A=√P(P-a)(P-b)(P-c));

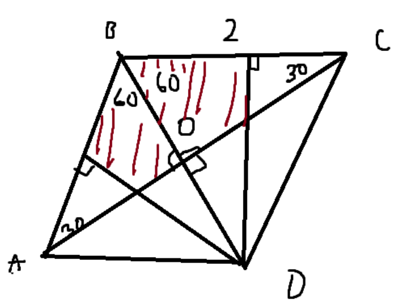
P=25+12+17=25+29=54/2=27;
√27*2*15*10)=√3^3*2*3*5*5*2)=√3^4*2^2*5^2)=3^2*2*5=90;
90*2=180;

<u>The area of all the points closest to point x:</u>

Problem 1:

> Rhombus $ABCD$ has side length 2 and $\angle B = 120°$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?
>
> **(A)** $\frac{\sqrt{3}}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{2\sqrt{3}}{3}$ **(D)** $1+\frac{\sqrt{3}}{3}$ **(E)** 2

Strategy: Construct a diagram and determine the area using common methods.



We can determine the area of the region shaded in red by determining the area of the triangle ABC and subtracting the two congruent triangles formed on the sides. The area of the larger region is equal to sqrt(3).

The area of one of the smaller congruent triangles is sqrt(3)/3; Subtracting the areas we get 2sqrt(3) as the final result.

<u>Pythagorean Theorem:</u>

Common Root Triples: (3,4,5), (5,12,13), (7,24,25), (8,15,17), (9,40,41), (11,60,61),(12,35,37)

<u>Typical Types of Problems:</u>

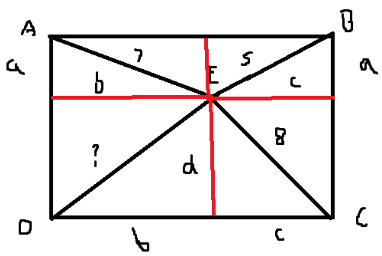
1. Shadow, ladder, 2d shape, traveling (Basic Problems)
2. Inside a 3d shape
3. Combining algebra and pythagorean theorem
4. Root Triple Problems
    a. The generalized form of primitive triples is: $2nm$, n^2-m^2, n^2+m^2 (where n,m are relatively prime and of different parity)

Combining Algebra and pythagorean's theorem

Problem 1:

29. ___________ units   Point E lies within rectangle ABCD. If AE = 7, BE = 5 and CE = 8, what is DE? Express your answer in simplest radical form.

Strategy: 1. Create a diagram, representing the necessary side lengths with variables. 2. Develop some equations with these variables, keeping the target in mind. 3. Solve for the target value.

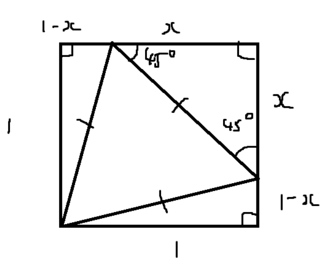


Target: $\sqrt{d^2+b^2}=?$

$a^2+c^2=25$; $a^2+b^2=49$; $d^2+c^2=64$; => $b^2-c^2=24$; $d^2+c^2=64$; => $b^2+c^2=88$; DE=sqrt(88);

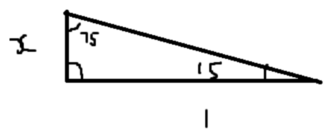
Problem 2: The area of the largest equilateral triangle that can be inscribed in a square of side length 1 unit can be expressed in the form a b − c units^2 , where a, b and c are integers. What is the value of a + b + c?

Method 1: Create a diagram of the largest possible equilateral triangle inside the square, representing the necessary side lengths with variables, and use the pythagorean theorem to create equations, and solve for the unknowns.



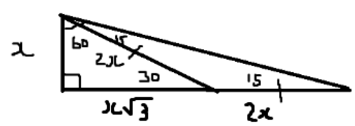
$2x^2=(1-x)^2+1$; => $2x^2=1-2x+x^2+1$; => x^2+2x-2; => $x=\sqrt{3}-1$ (the only positive solution to the equation);
$2*(4-2\sqrt{3})=8-4\sqrt{3}$
Area of equilateral triangle = $\sqrt{3}/4*(8-4\sqrt{3}) = 2\sqrt{3}-3$;
Thus, a+b+c=2+3+3=8;

Method 2: Create a diagram focusing solely on the triangle with legs 1 and 1-x in the previous diagram. Use trigonometry/special triangles to solve for the shorter leg.



tan(15)=x.

To determine the value of tan(15) we can construct an isosceles triangle with angles, 15,15, and 150.



By constructing the isosceles triangle we also form a 90,60,30 triangle such that we can represent the base in terms of x.

$x\sqrt{3} + 2x = 1;$ => $x = 1/(2 + \sqrt{3});$ => $x = 2 - \sqrt{3};$
Thus, $1 - x = 2 - \sqrt{3};$ => $x = \sqrt{3} - 1;$ $2x^2 = 8 - 4\sqrt{3};$ => $\sqrt{3}/4 * (8 - 4\sqrt{3});$ => $2\sqrt{3} - 3;$
Thus, a+b+c=2+3+3=8

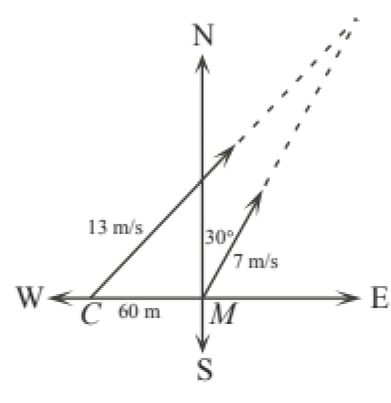
Problem 3:

4. A cat is located at C, 60 metres directly west of a mouse located at M. The mouse is trying to escape by running at 7 m/s in a direction 30° east of north. The cat, an expert in geometry, runs at 13 m/s in a suitable straight line path that will intercept the mouse as quickly as possible.

(a) If t is the length of time, in seconds, that it takes the cat to catch the mouse, determine the value of t.

(b) Suppose that the mouse instead chooses a different direction to try to escape. Show that no matter which direction it runs, all points of interception lie on a circle.

(c) Suppose that the mouse is intercepted after running a distance of d_1 metres in a particular direction. If the mouse would have been intercepted after it had run a distance of d_2 metres in the opposite direction, show that $d_1 + d_2 \geq 14\sqrt{30}$.



(a)

Let the horizontal distance the mouse travels be x and the vertical distance be xsqrt(3);
Thus, the horizontal distance the cat travels must be x+60 and the vertical distance must be xsqrt(3);

(x+60)^2+[x*sqrt(3)]^2=169*t^2
x^2+[x*sqrt(3)]^2=49*t^2
4x^2+120x+3600=169t^2
4x^2=49t^2
x=7t/2
49t^2+420t+3600=169t^2
2t^2-7t-60=0;
t=15/2;

(b)

Let Z(a,b) be the coordinate of the intersection of the lines; We know the lines ZM/ZC=7/13

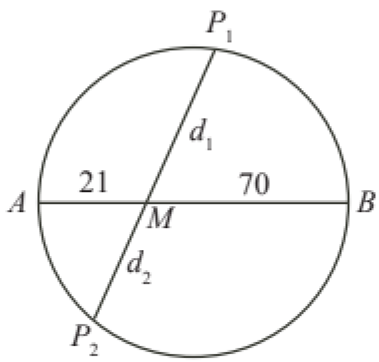
Thus,

$$\begin{aligned}
\frac{\sqrt{(x+60)^2+y^2}}{\sqrt{x^2+y^2}} &= \frac{13}{7} \\
\frac{(x+60)^2+y^2}{x^2+y^2} &= \frac{169}{49} \\
49((x+60)^2+y^2) &= 169(x^2+y^2) \\
0 &= 120x^2 - 2(49)(60)x - 49(60^2) + 120y^2 \\
0 &= x^2 - 49x - 49(30) + y^2
\end{aligned}$$

We can manipulate this to get: $(x-\frac{49}{2})^2+y^2=(\frac{91}{2})^2,$
Assuming coordinates is very useful when proving an analytical property

Thus, all points are on a circle with radius = 91/2 and center at (49/2,0)

(c)



The following is a circle we can produce as, from (b), the intersections of AP1 and d1lie on the same circle as that of AP2 and d2. Thus, by power of a point, $21*70=d1*d2$;

By AM-GM we know that:

$(d1+d2)/2>=\sqrt{d1*d2}$

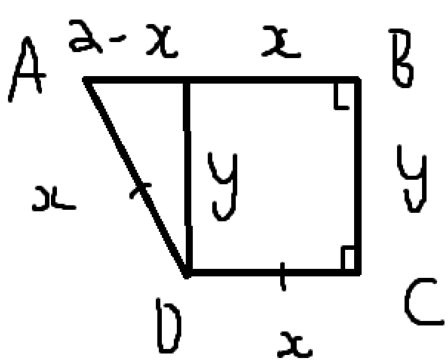
Thus.

$d1+d2>=14\sqrt{30}$

Problem 4:

For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB = 2$, and $CD = AD$. How many different values of $p < 2015$ are possible?

(A) 30 **(B)** 31 **(C)** 61 **(D)** 62 **(E)** 63



$x^2-y^2=(2-x)^2$

$y^2-4x+4=0$

$y^2=4(x-1)$

P = 2+y+2x = 2 + 2sqrt(x-1) + 2x < 2015
Thus, 1^2+1<=x<=31^2+1; x has 31 valid values.
Each valid value of x has a corresponding valid value of y, thus there are 31 possible values for p.

Root Triple Problems:

Problem 1: (Math challengers provincial question 26)

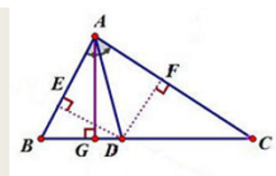
Bisector Theorem:

Properties:

1. Basic Formula: a/b=c/d
2. Perpendicular Connector Equality
3. Constructing Parrell Lines

Problem 1:

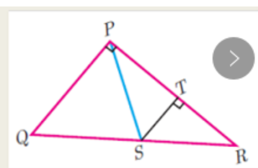
for example, AB=6, AC=10, BC=14, AD bisects angle A and intersects with BC at D. What is the distance from D to AB? and the length of GD?



From **Perpendicular Connector Equality** we know that DF = ED, thus the are can represent the are as 3x + 5x = 8x; From heron's formula: Area = sqrt(15*1*5*9) = 15sqrt(3). Thus, ED = 15sqrt(3)/8; GD can be solved using the definitional formula as well as pythogoreans theorem.

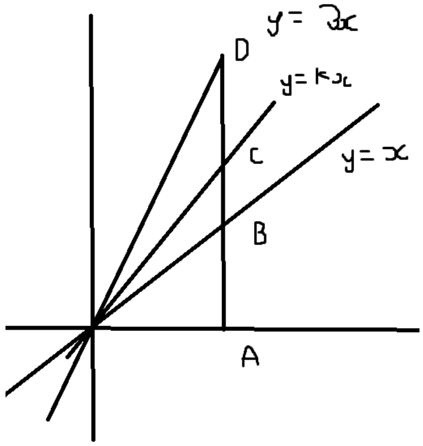
You can also use the **Perpendicular Connector Equality** to prove basic relationships:

2. In $Rt\triangle QPR$, $\angle QPR = 90°$, PS is its bisector. If $ST \perp PR$, prove that $ST \cdot (PQ + PR) = PQ \cdot PR$



ST * PR + ST * PQ = PQ * PR; and ANS is easily derived

Problem 3:

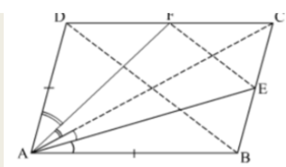


When provided with multiple lines and their slopes it's common to form a combined triangle and exploit the nature of their slopes/trig ratios with pythag and common ratios.

Here, we can use ratios + pythag to solv.

Problem 4:

3. In the quadrilateral ABCD, AB=AD, the bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that EF//BD.



To prove parallel lines, there are multiple methods. One of these methods is using proportions (/proving similar triangles). In this problem, if we prove EC / BE = FC / DF, then the lines must be parrallel.

From the angle bisectors, we know that DF/FC = AD / AC and that BE / EC = AB / AC; We also know that AB = AD, thus, DF / FC = BE / EC and DB is parallel to FE!

<u>Triangles Properties:</u>

<u>Properties of a triangle: (Let the sides of a random triangle be a,b,c)</u>

1. Sum of sides: **$a+b>c$ and $a-b<c$.** If these two inequalities are not satisfied then the shape is not a triangle and no angles can lead to its formation.
2. If a triangle contains a 90 degree angle (is a right triangle), then the sides must satisfy: $a^2+b^2=c^2$ where $c>a,b$. If this is not satisfied, then the shape is not a right triangle.
3. If a triangle is acute then the sides satisfy the following inequality: $a^2+b^2>c^2$.
4. If a triangle is obtuse then the sides satisfy the following inequality: $a^2+b^2<c^2$.

<u>Problems:</u>

Problem 1:

8. How many integers N are there such that N, 15, and 14 are the sides of an acute triangle? An acute triangle is a triangle whose angles are all less than $90°$?

Strategy: Implement the properties of the sides of a triangle we discussed to solve for the range of N.

The triangle is acute. Therefore, if N<15, then 14^2+N^2>15^2 **N>sqrt(29)>=6** and, if N>15, then 14^2+15^2>N^2, so **N<sqrt(421)<=20.**

Next, we must check if all of the values within our current range of N satisfy the triangle side inequality.

Implementing the triangle inequality theorem we get the following: 29>N>1. This range is less restrictive than our original range so we can use the original as our final range.

Because 6<=N<=20 there are a total of 15 possible values of N.

Problem 2:

Let T_1 be a triangle with side lengths $2011, 2012,$ and 2013. For $n \geq 1$, if $T_n = \triangle ABC$ and $D, E,$ and F are the points of tangency of the incircle of $\triangle ABC$ to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths $AD, BE,$ and CF, if it exists. What is the perimeter of the last triangle in the sequence (T_n)?

$\textbf{(A)}\ \frac{1509}{8} \qquad \textbf{(B)}\ \frac{1509}{32} \qquad \textbf{(C)}\ \frac{1509}{64} \qquad \textbf{(D)}\ \frac{1509}{128} \qquad \textbf{(E)}\ \frac{1509}{256}$

This is a problem in which "experimentation" is VERY useful.

We find the following side lengths for the first 4 terms of Tn:
2011,2012,2013
1005,1006,1007
502,503,504
501/2, 503/2, 505/2

Now, we see that the middle term is always halved, and the others are just 1 greater and 1 less.
Now, we can use the property that c>S/2 and see when this is satisfied

503/2^n + 1 < 1509/2^(n+1)
After some testing, it becomes clear that n=8 violates the inequality.
Thus, n=7, or a triangle with a perimeter of **1509/128** is the last triangle in the sequence.

Cyclic Quadrilaterals:

Properties of cyclic quadrilaterals:

1. Sum of opposite angles in a cyclic quadrilateral is always 180.
2. When we draw the diagonals of a cyclic quadrilateral, we form four pairs of equal inscribed angles

Proving cyclic quaderlaterlas:

1. Sum of two opposite angles in a quadralteral = 180; (this implies that the other two sides also have sum=180)
2. Proving that four points are concyclic (lie on the same circle): If points C and D lie on the same side of segment AB such that angle ACB = angle ADB, then the four points A,B,C, and D are concclic.

Ptolemy's Theorem:

1. Ptolemy's Theorem states that in cyclic qudrilateral ABCD, with a=AB, b=BC, c=CD, d=DA, and diagonals e and f, we have: ac+bd=ef

Problems:

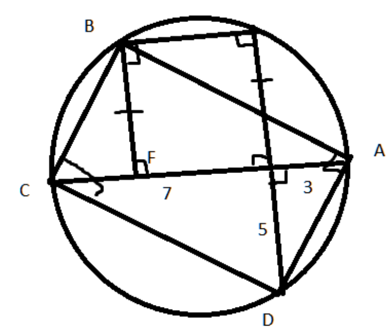
Problem 1:

We know that DC=9/2 from similar triangles and that AD=2*BC;

Thus, by ptolemy's theorem, BC^2 * 2 = 110-27;

Thus, we solve for BC and sub back into the equation to solve for AD.

Problem 2: In the figure, ABCD is a quadrilateral with right angles at A and C. Points E and F are on AC, and DE and BF are perpendicular to AC. If AE=3, DE=5, and CE=7, then find BF.

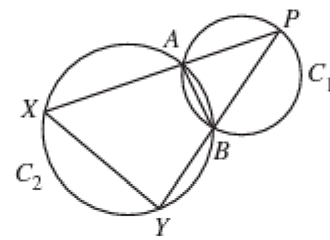


We extend the line DE to X such that we form a rectangle BFEX.

By power of a point XE*5=21; Thus, XE=21/5; BF=XE=21/5;

Problem 3:

3. (a) In the diagram, the two circles C_1 and C_2 have a common chord AB. Point P is chosen on C_1 so that it is outside C_2. Lines PA and PB are extended to cut C_2 at X and Y, respectively. If $AB = 6$, $PA = 5$, $PB = 7$ and $AX = 16$, determine the length of XY.



(b) Two circles C_3 and C_4 have a common chord GH. Point Q is chosen on C_3 so that it is outside C_4. Lines QG and QH are extended to cut C_4 at V and W, respectively. Show that, no matter where Q is chosen, the length of VW is constant.

(a): Just from inspection, APB and XYP look similar. Of course, this is not for sure; but it can guide our search.

We already know triangles XYP and ABP share an angle P; so we need one more common angle.

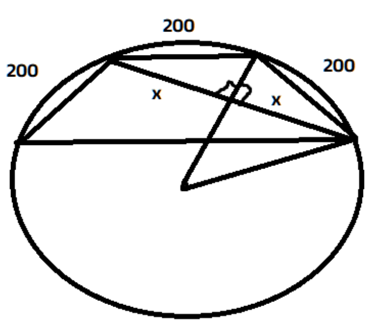
We know <ABP=180-<ABY; We also know <AXY=180-ABY; Thus, <AXY=<ABP. This is the common angle we were looking for. Thus, triangle APB is similar to triangle XYP.

Thus, PB/XP=AB/XY; 7/21=6/XY; **XY=18**;

(b): If we can prove that <XBY is constant, then XY must be constant as well. Because AB is a constant chord, <BXP and <APB are both constant (no matter the location of P). We know that <XBY=180-<XBP. Thus, <XBY=<BXP+<BPX. Thus, <XBY is constant and XY, as a consequence, is also constant.

Problem 4:

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?



Our goal is to solve for x such that we can use ptolemy's inequality.

The area of a triangle formed by the base 200 and the two radi can be solved for with pythagoreans theorem.

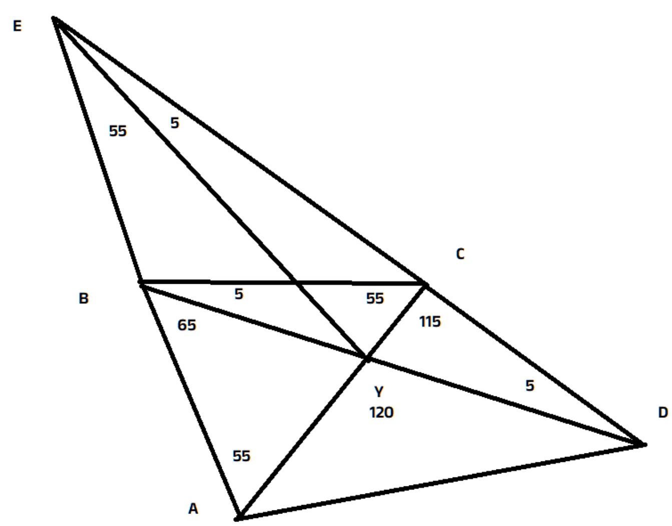
The height is 100sqrt(7)

The area is 10000sqrt(7)
200sqrt(2)*x=20000sqrt(7)
x=100sqrt(7)/sqrt(2)
Now we can solve for the product of the diagonals and use ptomely's theorem to finalize.

Problem 5:

Quadrilateral $ABCD$ has $AB = BC = CD$, angle $ABC = 70$ and angle $BCD = 170$. What is the measure of angle BAD?

$\textbf{(A)}\ 75 \qquad \textbf{(B)}\ 80 \qquad \textbf{(C)}\ 85 \qquad \textbf{(D)}\ 90 \qquad \textbf{(E)}\ 95$



We can make the following construction and manipulate the properties of cyclic quadralateral to form the diagram above. Then, we abuse the fact that EY=AY=YD. Thus, mYAD = mYDA = 30.

Angle Problems:

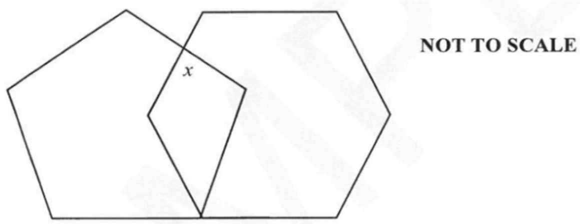
Types of Problems:

1. 2d Shape Angles
2. Using parallel line angle theorem

2d Shape Angles:

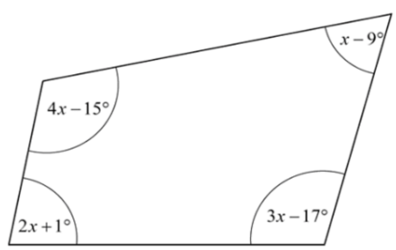
Problem 1:

A regular pentagon and a regular hexagon overlap as shown.
Find the angle x. Show your reasoning clearly.



Target: Angle x;
Strategy: Determine all angles within the intersection quadrilateral excluding x, and subtract their sum from 360;
Pentagon interior angle = 108; Hexagon interior angle = 120;
Bottom angle = 108 + 120 - 180 = 48 ; (Similar to set theory).
Quadrilateral total = 360;
X = 360 - (108 + 120 + 48) = 84;

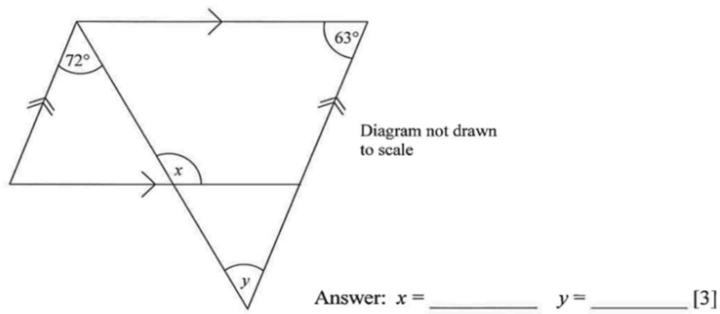
Problem 2:



Target: x;
Strategy: Use the fact that the sum of the interior angles of quadrilateral is 360 to solve for x;
4x -15 + x-9+2x+1+3x-17=360; => 10x=400; => x=40;

<u>Using Parallel Line Angle theorem:</u>

Problem 1:



Target: x and y;

Strategy: Using parallel line angle relations to solve for both variables;

If you flip the shape by 90 degrees to the right, you can clearly see that 2 additional parallel lines are present. 72 and y are alternating angles, such that y=72; The rightmost interior angle of the triangle outside the parallelogram is a corresponding angle, such that the rightmost interior angle of the triangle is 63 degrees. X is the exterior angle of the triangle such that x= 72+63 = 135;

Similarity and Congruence Problems:

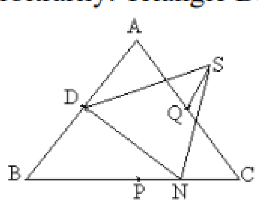
Types of Problems:

1. Congruence (4 proofs):
   a. SAS => Two sides and the angle in between
   b. SSS => All three sides are equal
   c. ASA => Two angles and one side (angles denote similarity and side denotes congruence)
   d. RHS => Hypotenuse and any leg of a right triangle or equal
2. Similarity Problems

Congruence:

Problem 1:

1. In equilateral ΔABC, D, Q, P are three midpoints on AB, BC, CA respectively, point N is on PC arbitrarily. Triangle DNS is the other equilateral △. Prove $PN = QS$.



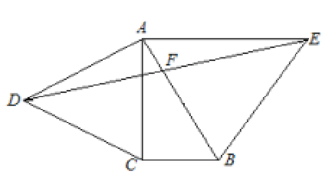
Strategy: Construct a triangle that is congruent to triangle DPN which has known sides.

We can connect DP and QD. We know DN and DS are equal, and we also know DQP is an equalateral triangle and thus, DQ = DP. We also know that 60+angleDQS=60+anglePDQ, so angleDQS=anglePDQ.

Thus, by SAS proof, triangles DQS and DPN are congruent and have sides PN and QS equal.

Problem 2:

2. In right angled triangle ABC with angle A being 30^0 two equilateral triangles ACD, ABE are draw on AB and AC respectively, Join ED and ED intersects AB at F. Prove F bisects ED.



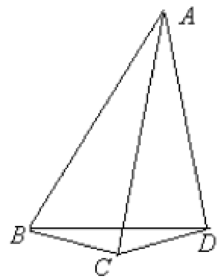
Strategy: construct congruent triangles that relate to the target (in this case the relationship between FE and DF).

Construct the height of equalateral triangle AEB about base AB. Let the height intersect with base AB at O. EOB is congruent to ABC by ASA. Thus, OE = AC = AD. Also, we know DAF = EOF = 90 and EFO = AFD by opposite angle theorem.

Thus, triangles OEF and ADF are congruent and FE = DF. So F bisects side DE.

Problem 3:

4. Given $\angle ABD = \angle ACD = 60^\circ$, $\angle ADB = 90^o - \frac{1}{2}\angle BDC$. Prove ΔABC is an isosceles triangle.



Strategy: Use angle chasing to prove that angles ACB and ABC are equal.

Let O equal the intersection of BD and AC. We can angle chase to know that OCD is similar to AOB. From that, we also know that AOD is similar to BOC. Thus, angle ACB = angle ABC and triangle ABC is isoceles.

Problem 4: Trapezoid ABCD has right angles at A and D, and diagonals AC and BD intersect at point E. The area of ABE is 25 units squared , and the area of DEC is 49 units2 . If AD = 6, what is the area of trapezoid ABCD?

Target: Area of trapezoid;
Strategy: Create a diagram to better interpret the question. Then use the ratio between the areas to find the ratio between the heights. Then use the total height of 6 to determine the triangle's respective heights. Now solve for both bases and calculate the area.

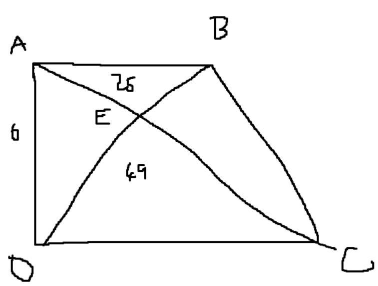


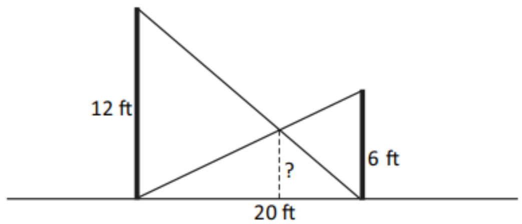
Diagram:

Method 1: (Using similarity)

Side length ratio = √25:49)=5:7;
Total height =6. Therefore height of ABE = 2.5; height of EDC = 3.5;

Base of ABE = 20; Base of EDC = 28;
Total Area = (20+28)/2 * 6; = 3 * 48 = 144;

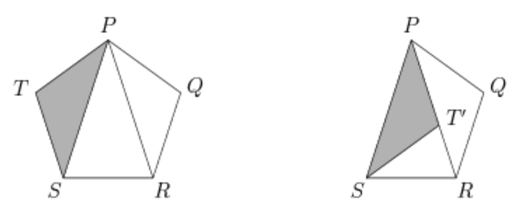
Problem 5: Two vertical poles with heights 6 ft and 12 ft, respectively, are placed 20 ft apart. A wire is strung from the top of each pole to the base of the other pole. How high above the ground do the two wires cross?



Target: Length of the height of the intersection point;
Strategy: Determine the Area of the smaller triangle through determining their respective heights. Afterwards, we can determine the length of the height.

Side ratio = 12:6=2:1;
Heights = 20/3 and 40 /3 respectively;
Area of the smaller triangle = 20/3 * 6 * ½ = 20;
Area of triangle with height 6 and base 20 = 20 * 6 * ½ = 60;
Area of lowest triangle of base 20 = 60 -20 = 40;
Height: 20 * h * ½ = 40; => h=4 ft;

Problem 6: Similarity (Specialized Toward Pentagons):



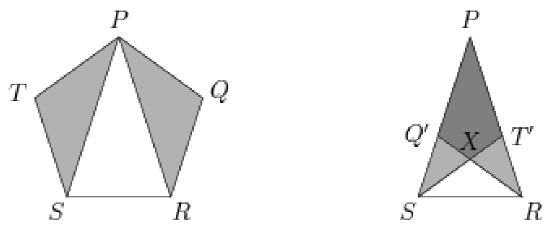
Find the ratio $\frac{PT'}{T'R}$. Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

If we angle chase, we can determine that triangle ST’R is similar to triangle PSR. Thus, T’R/SR=SR/PR. To avoid getting stuck here, one must acknowledge that SR = ST’ = PT’ and that PR = PT’ + T’R.

Thus, T’R/PT’=PT’/(PT’+T’R). We can let T’R=x and PT’ = y;
Thus, x/y=y/(x+y) and y/x=1+x/y. We can let y/x=z such that:
z=1+1/z; => z^2-z-1 = 0; => z = ANS = [1+sqrt(5)]/2

Problem 7: (An extension of problem 6):

(c) Regular pentagon $PQRST$ has an area of 1 square unit. The pentagon is folded along the diagonals SP and RP as shown on the right. Here, T' and Q' are the positions of vertices T and Q respectively, after the foldings. The segments ST' and RQ' intersect at X.



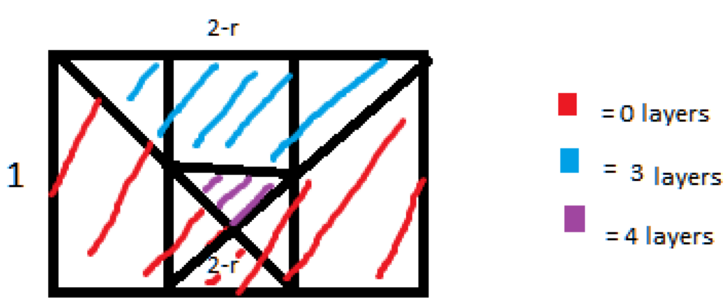
Determine the area (in square units) of the uncovered triangle XSR. Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

Here we can use similarity along with base-height proportionality to help solve this problem. We know that PST'/ST'R = PT'/T'R. Thus,

Folding Problems:

Problem 1:

8. A rectangular sheet of paper, $ABCD$, has $AD = 1$ and $AB = r$, where $1 < r < 2$. The paper is folded along a line through A so that the edge AD falls onto the edge AB. Without unfolding, the paper is folded again along a line through B so that the edge CB also lies on AB. The result is a triangular piece of paper. A region of this triangle is four sheets thick. In terms of r, what is the area of this region?

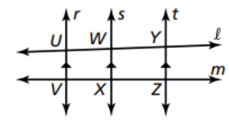


The purple region is inside of a square of side length = 2-r; Thus, purple region = **¼(2-r)^2**

Theorems:

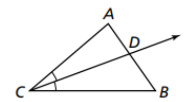
1. **Converse of the Triangle Proportionality:** Theorem If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

2. **Three Parallel Lines Theorem:** If three parallel lines intersect two transversals, then they divide the transversals proportionally.



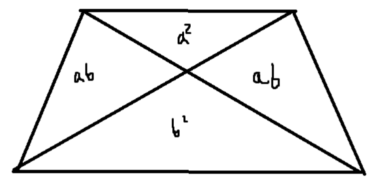
UW VX

3. **Triangle Angle Bisector Theorem:** If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides.

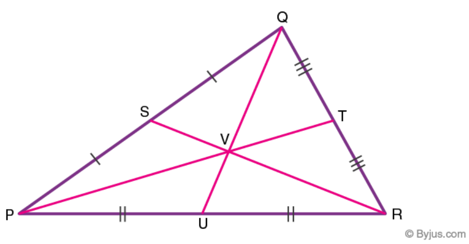


$$\frac{AD}{DB} = \frac{CA}{CB}$$

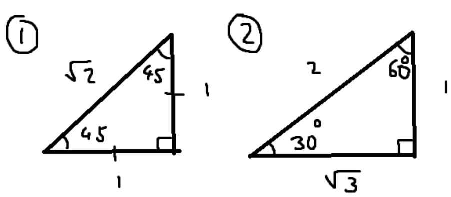
4. **Opposite Triangle Trapezoid Theorem:** If the areas of two opposite similar triangles are given, then the area of the two congruent remaining triangles is equal to $\sqrt{Area1 * Area2}4$



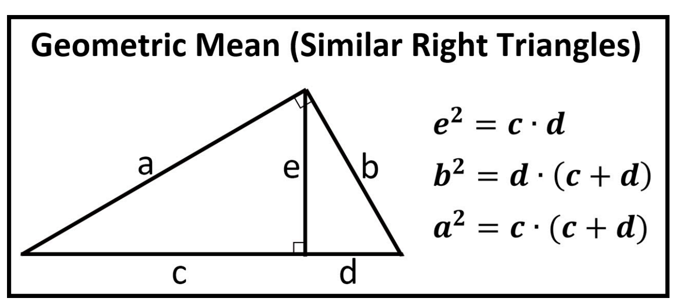
5. **Triangle Centroid Ratio**: The centroid theorem states that the centroid of the triangle is at 2/3 of the distance from the vertex to the mid-point of the sides. -> QV = 2/3 QU, PV = 2/3 PT and RV = 2/3 RS



6. **Special Triangles:** There are a two common types of special triangles:



7. **Altitude geometric mean theorem:**



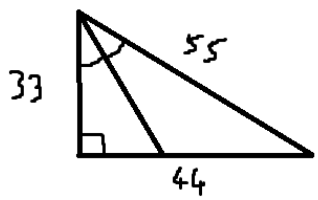
Bisector Proportionality

Problem 1:

25. ____________ cm A line bisecting the larger acute angle in a triangle with sides of length 33, 44 and 55 cm divides the opposite side into two segments. What is the length of the shorter segment of that side? Express your answer as a common fraction.

Strategy: Use bisector proportionality theorem to solve for the unknown length

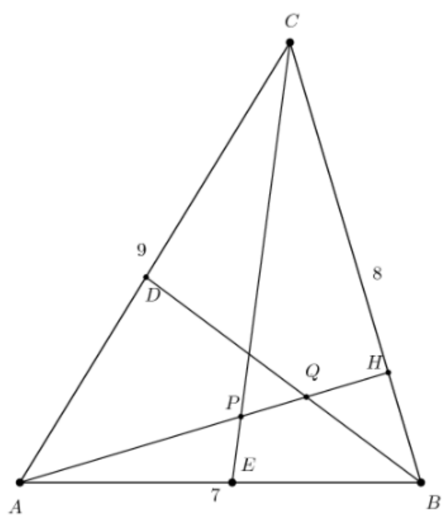
We can see that the triangle given is a right triangle given the 33, 44, 55 satisfying the 3,4,5 triples. We can construct a diagram to better understand the problem given:



We can represent the shorter side with x, and the longer side with 44-x and set up the ratio:
x/44-x=33/55 => x/44-x=3/5 => x=33/2

Problem 2:

In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?



(A) 1 **(B)** $\frac{5}{8}\sqrt{3}$ **(C)** $\frac{4}{5}\sqrt{2}$ **(D)** $\frac{8}{15}\sqrt{5}$ **(E)** $\frac{6}{5}$

Solution:

We can determine the length of side AH by determining the total area of the triangle and dividing it by 4 .
The total area of the triangle = sqrt(12*3*4*5) = 12sqrt(5). Dividing by 4, we can determine that AH is equal to

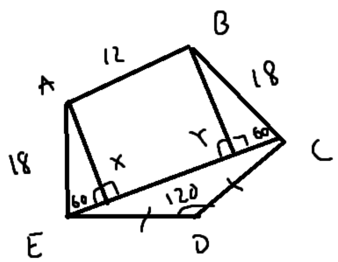
3sqrt(5). We can determine the length of CH through pythagoras theorem. CH=sqrt(81-45)=sqrt(36)=6; HB=2;

Special Triangles

Problem 2 (special triangles):

In pentagon ABCDE, ∠E and ∠C are right angles and $m\angle D = 120°$. If AB = 12, AE = BC = 18 and ED = DC, what is ED? Express your answer in simplest radical form.

Strategy: Construct a relatively accurate diagram to better understand the problem, prove that AB is parallel to EC and use special triangles to solve for the length ED.

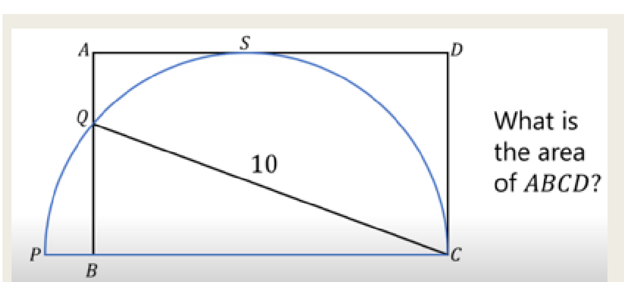


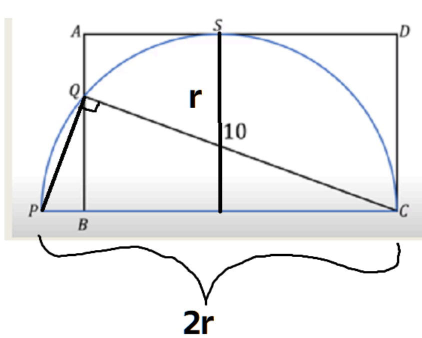
AB is parallel to EC given that the triangle with hypotenuse AE is congruent to the triangle with hypotenuse BC. This is because the longer legs of both triangles are equal and parallel thus, AB is also parallel to EC. Calculating the length of the smaller leg of the congruent triangles: 18/2=9;

Given that AB is parallel to EC, we know that AB =XY such that EC=9*2+12=30; Dividing triangle EDC into two congruent triangles by connecting point d to the base, constructing a perpendicular bisector to EC. By doing this, we construct a 30-60-90 triangle with a side 30/2=15 corresponding to root(3). Thus, ED = (15/root(3)) * 2 = **10 * root(3)**

Geometric Mean Theorem:

Problem 1:





We connect P to Q such that <PQC = 90 degrees. Then, we use the geometric mean theorem to see that 10^2 = PC * BC; Thus, BC = 50/r;

We know that DC = r, thus, the area of rectangle ABCD = 50/r * r = 50.

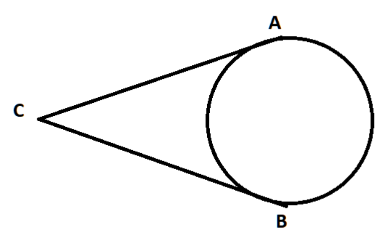
Circle Geometry:

Formulas:
Area = πr^2;
Circumference = $2 r \pi \parallel d \pi$;
Area of sector = $\pi r^2 * n/360$; or (L * r) / 2
Diameter = 2 r;
Pi = 3.14159;

Power of a Point Theorem:

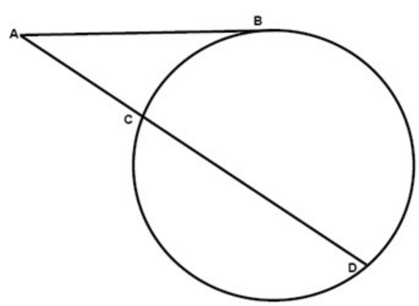
Power of a point formulas:

1. **Two tangents from a point**. Two tangents from the same point to a circle are always equal.

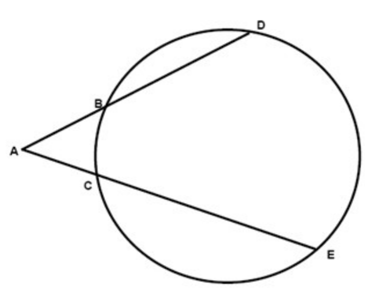


AC=BC

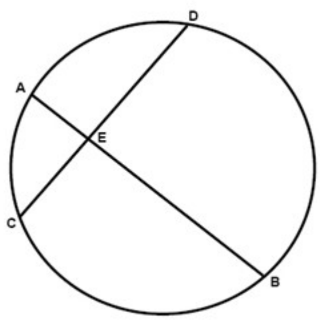
2. **A tangent and a secant from a point.** Given tangent AB and secant AD at left, we have: **AB^2=AC(CD)**



3. **Two secants from a point.** Given secants AD and AE at right, we have: **(AB)(AD)=(AC)(AE):**



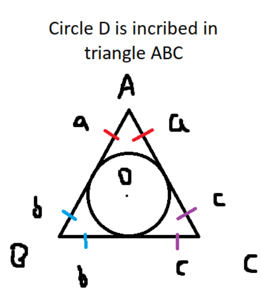
4. **Two chords through a point.** Given two chords AB and DC which intersect at E, we have (AE)(EB)=(ED)(EC).



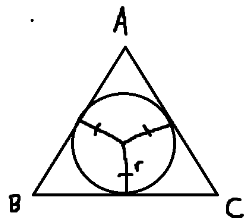
Properties:

1. Equal arcs subtend equal angles and vice versa.
2. Equal chords are equidistant from the center and vice versa.
3. The perpendicular bisector of a chord passes through the center of the circle
4. If two chords are perpendicular then their hypotenuse is the diameter
5. The line connecting the intersection points of two circles is perpendicular to the line connecting their centers.
6. The degree measure of an inscribed angle is half the measure of the intercepted arc.
7. The degree measure of vertical angles formed by two chords intersecting inside a circle is half the sum of the measures of their intercepted arcs.
8. **Quarter Circles**: Common strategy involves expanding to regular circle and drawing chords to implement chord theorem.
9. **Parallel lines** can be used to draw isoceles trapezoids

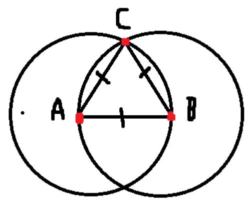
Common Positions:

1. 

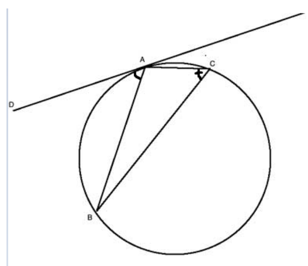
AC + BC - AB = 2c;



2. Area = ½*r*(AB + BC + AC)

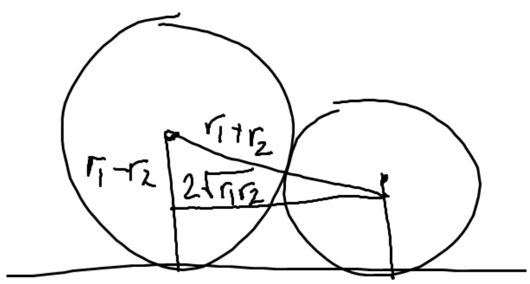


3. ABC is equilateral

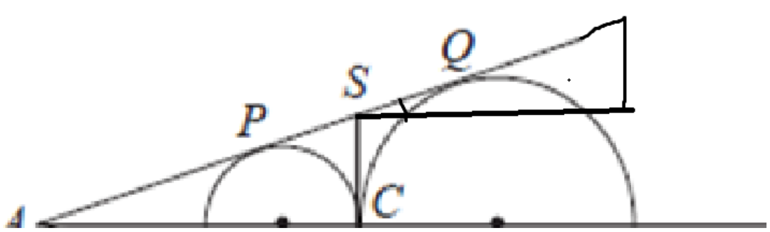


4. Angle DAB = angle ACB (tangent-chord theorem) ->

The Tangent-Chord Theorem states that the angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord: ∠BAD ≅ ∠BCA



5. let r1,r2 be the respective radii



6.

Common Strategy when presented with adjacent circles and similar triangles.

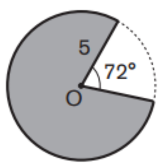
Typical Types of Problems:

1. Basic Implementation of Formulas;
2. Using common properties and positions;

<u>Basic Implementation of formulas:</u>

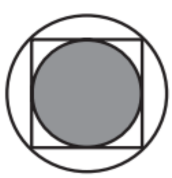
Problem 1: What is the circumference of a circle that has an area of 289π mm? Express your answer in terms of π.

Target: Circumference;
Strategy: Use common formulas to create an equation and solve for the unknown value;
$\pi r^2=289\pi$; r=17; d=r*2=34; circumference = 34 π;
Problem 2: Circle O has a radius of 5 meters. When a 72° sector is removed from circle O, as shown, what is the area of the shaded region that remains? Express your answer in terms of π.



Target: Shaded Sector;
Strategy: Determine the angle of the shaded sector and solve using the formula;
Shaded Sector Angle = 360 -72 = 288;
Total Area = 25π;
Shaded Area = 288/360 * 25π = ⅘ * 25π = 20π;

Problem 3: The figure shows a circle inscribed in a square that is inscribed in a circle. If the larger circle has area 256π mm2, what is the radius of the smaller, shaded circle? Express your answer in the simplest radical form.



Target: Radius of Shaded Circle:
Strategy: Solve in steps, solving for the necessary properties, starting from the largest shape and slowly moving down.
Large Circle: $\pi r^2=256\pi$; r = 16; d=32;
Square: s=√32/2)=4;
Small Shaded Circle: r = d/2; d= 4; r = 2;

<u>Using common properties + common positions</u>

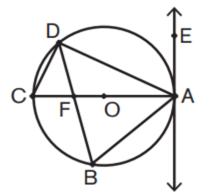
Problem 1:

235.________° What is $m\angle ABD$?

236.________° What is $m\widehat{AB}$?

237.________° What is $m\angle BAE$?

238.________° What is $m\angle CFD$?



$\overleftrightarrow{AE}$ is tangent to circle O

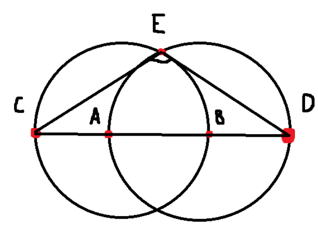
$\overleftrightarrow{AE} \perp \overline{AC}$

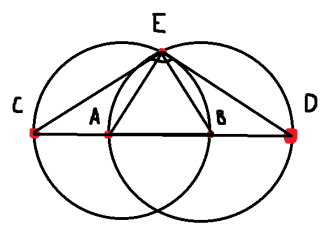
$m\angle BDC = 40°$

$m\widehat{AD} = 125°$

1. ABD = AOD / 2; ABD = 125 / 2 = 62.5 degrees;
2. AC is the diameter of the circle; Thus, CD and AD must be perpendicular; CDA = 90; ADB = 90 - CDF; CDF = 40; ADB = 50; AOB = 2 * ADB = 2 * 50 = 100 degrees;
3. CA is perpendicular to EA. Thus, CAE = 90 degrees; CDB = 40 ; CBO = 40 * 2 =80; CBA = CBO /2 = 40; BAE = 90 + 40 = 130 degrees;
4. CDF = 40; DCA = AOD / 2 = 125 / 2 = 62 . 5; CFD = 180 - 40 - 62.5 = 77.5 degrees;

Problem 2: Two congruent circles centered at points A and B pass through the other circle's center. The line containing both a and B is extended to intersect the circles at points C and D. The circles intersect at two points one of which is E. What is the degree measure of angle CED?

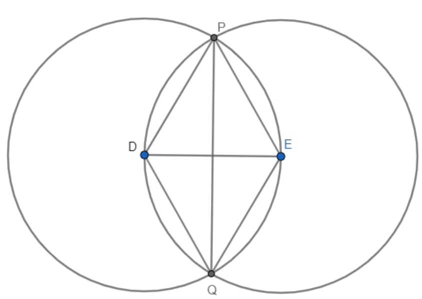




=> We can draw an equilateral triangle by connecting AE and BE because the sides of the triangle are all the radius.

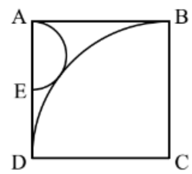
Triangles CEA and DEB are congruent because EB, EA, CA, and BD are all equal to the radius and because the angles connecting these sides are both equal to 180 - 60 because of the equilateral triangle.
Because both triangles CEA and DEB have 2 equal angles, they are both isosceles triangles in which angles CAE and EBD are equal to 180-60=120. Because the triangles are isosceles, the two other angles are equal, such that they are equal to (180-120)/2=30; Thus, Angles CEA and BED are both equal to 30 such that angle CED = 120 degrees;

Problem 3: Circle D intersect the center of circle E, and circle E intersects the center of circle D. The radius of each circle is 6 cm. The area of the shaded region where the circles overlap can be expressed in simplest radical form as aπ + b c cm2. What is the value of a + b + c?



We have *DP* = *PE* = *EQ* = *QD* = *DE* so triangles *DPE* and *DQE* are equilateral and angles *PDQ* and *PEQ* are both 120 degrees. Thus the two sectors each have an area 1/3 that of one of the circles so together they have area (2/3)pi(6)^2 = 24pi. We have *PQ* = 6srqt(3) so the area of the kite is (1/2)(6)(6√3) = 18√3). The area of the shaded region is thus 24pi - 18√3.
The sum $a + b + c$ = 24 + (-18) + 3 = 9.

6. ____________ cm In square ABCD, shown here, sector BCD was drawn with a center C and BC = 24 cm. A semicircle with diameter AE is drawn tangent to the sector BCD. If points A, E and D are collinear, what is AE?



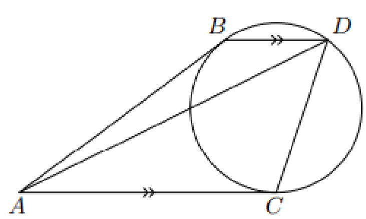
Problem 4:

To solve problems involving the tangency of two circles, the solution is often produced through the connection of the two circles' centers. This statement is reflected well in the following problem.

By connecting the centers of the two circles we produce a right triangle with sides of length 24, 24-r, 24+r; where r is equal to the radius of the smaller circle. After reducing this equation and combining like terms we get 96r=576. This, r=6;

AE=2r=12.

B4 In the following diagram, two lines that meet at a point A are tangent to a circle at points B and C. The line parallel to AC passing through B meets the circle again at D. Join the segments CD and AD. Suppose $AB = 49$ and $CD = 28$. Determine the length of AD.

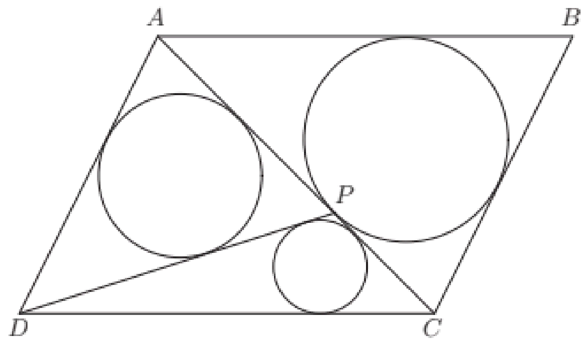


Problem 5:

AB and AC are tangent to the circle
Thus, AC=AB=49
Because of BD // AC, if we connect B and C to form line BC, angle ACB = angle CBD.
Because AC is tangent to the circle, by the tangent-chord-theorem, angle ACB = angle BDC. Thus, BC=CD=28 and triangle ACB is similar to triangle CDB (because they have equal angles).
Thus, AC/BC=BC/BD; => 49/28=28/BD; => BD=16
Let E = the midpoint of BD; ED=BD/2=16/2=8;
Because triangle BDC is isosceles, EC is its altitude;
sqrt(CD^2-ED^2)=EC; sqrt(28^2-8^2)=EC; => EC=12sqrt(5);

Because EC is the altitude, angle CED = 90 degrees;
Because AC // BD, angle ECA = angle CED = 90 degrees;
Let F be a point such that CFD = 90 degrees and FDB = 90 degrees;
Thus, quadrilateral EDFC will form a rectangle with sides ED=CF=8 and EC = DF=12sqrt(5);
Triangle AFD is a right triangle with legs of length AC+CF=49+8=57 and FD=12sqrt(5);
Thus, AD=sqrt((AC+CF)^2+FD^2)=sqrt(57^2+(12sqrt(5)^2)=63

Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P.



(a) (2 marks) Prove that $DA + AP = DC + CP$.

(b) (4 marks) Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides. A circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

(c) (4 marks) Suppose $DA + DC = 3AC$ and $DA = DP$. Let r_1, r_2 be the two radii defined in (b). Determine the ratio r_1/r_2.

Problem 6:

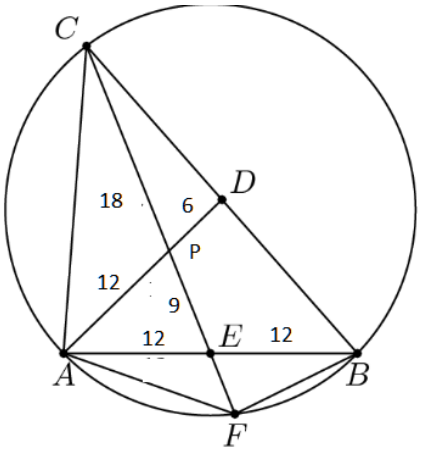
(a) We know that AB=DC=AP+x and that AD=BC=PC+x
Thus, (PC+x)+AP=(AP+x)+PC; Thus, DA+AP=DC+CP

(b) We know that AP/PC=[ADP]/[DPC]; We also know that DA+AP+PD=DP+PC+DC=y;
[ADP]=r1*½*(y)
[DPC]=r2*½*(y)
Thus, r1/r2=[ADP]/[DPC]=AP/PC;

(c) Let s=DA+AP=DC+CP and let x=AP, y=PC; Thus, DA=s-x; DC=s-y; We know that (s-x) + (s-y) = 3(x+y); Thus, s=2(x+y); Thus, AD=x+2y; DC=2x+y, DP=x+2y; Now we draw/imagine the height of triangle ADP;
By Pythagorean we have: $(2x+y)^2 - (x/2+y)^2 = (x+2y)^2 - (x/2)^2.$ Thus, x/y=4/3

Problem 7:

In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.



Here we use the median properties to determine the exact values of CP, PE and PD, PA;

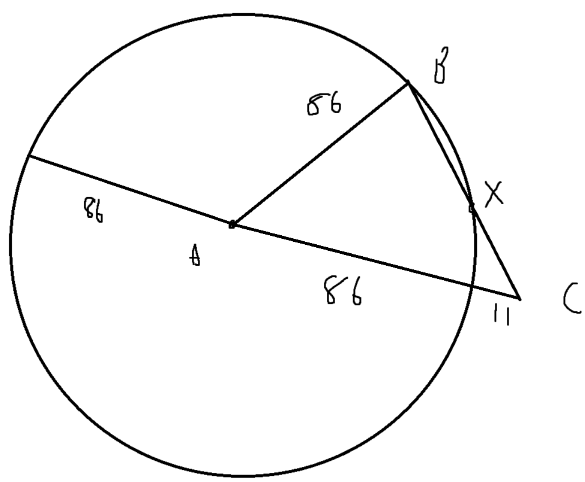
From this, we can use Power of a Point to determine EF=144/27=16/3
From this, we can use equal heights area ratios to find that [AEP]/[AEF]=9/(16/3);
We can determine the area of AEP through heron's formula. Thus, we can solve for AEF using the ratio we have predetermined.

Problem 8:

Problem

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 28 \qquad \textbf{(C)}\ 33 \qquad \textbf{(D)}\ 61 \qquad \textbf{(E)}\ 72$



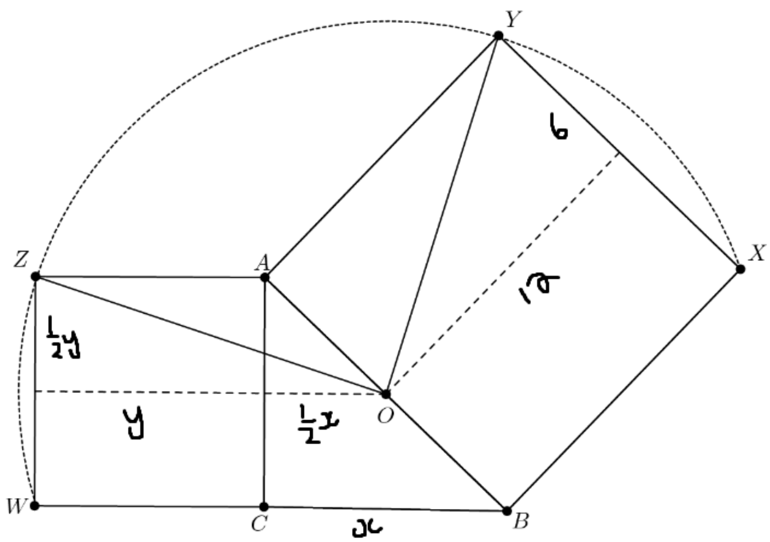
Drawing the diagram:

Once the diagram is drawn, it becomes obvious that CX * CB = 11*183 from Power of Point.

Problem 9:

In $\triangle ABC$, $\angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

$\textbf{(A)}\ 12+9\sqrt{3} \qquad \textbf{(B)}\ 18+6\sqrt{3} \qquad \textbf{(C)}\ 12+12\sqrt{2} \qquad \textbf{(D)}\ 30 \qquad \textbf{(E)}\ 32$



<u>The key fact that solves this problem is:</u> The center of a triangle will always be able to form a perpendicular bisector of any chord. Thus, to find the center of a triangle, we can draw the perpendicular bisectors of all the chords and then determine their intersection.

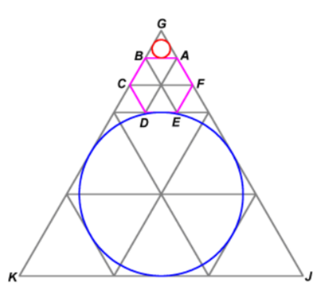
Now we have the equations:

(1/2y)^2+(y+1/2x)^2=180
=> 5y^2+x^2+4xy=720
5x^2+5y^2=720
4x^2-4xy=0
Thus, x=y; Thus, **perimeter is 12/sqrt(2) * 2 + 12 or ©**

Problem 10:

Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle?

(A) 18 **(B)** 27 **(C)** 36 **(D)** 81 **(E)** 108



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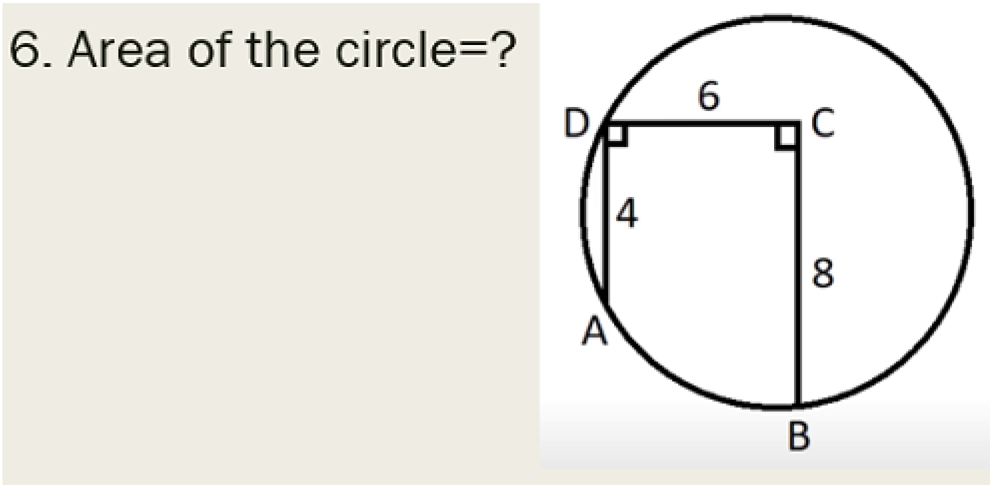
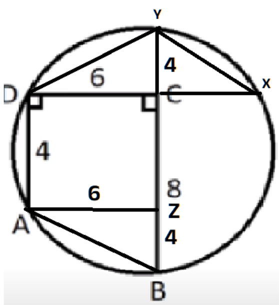
Problem 11:





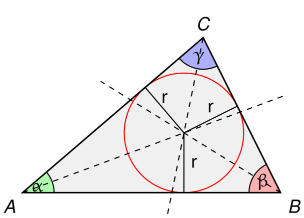
First, we notice that BC and DA are parallel. From this, we can extend BC and prove that YC = BZ = CZ = 4;

Now, we know that YC / CX = DC / BC due to similar triangles. Thus, 4/3 * 4 = XC = 16/3; Thus, we can solve for the radius by connecting AX and dividing by two.

<u>Inscribed Circles</u>

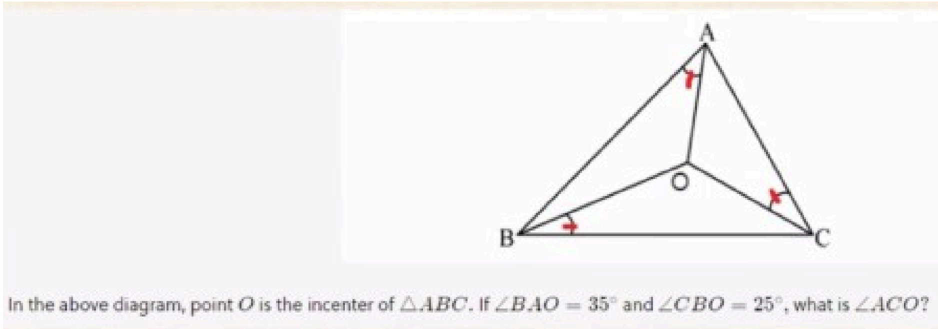
<u>Properties:</u>

1. The incenter is the intersection point of the angle bisectors of (a triangle)
2. The incenter is equidistant to each sides, equal to the inscribed circle's radius ®
3. The circle is tangent to each side



<u>Problems:</u>

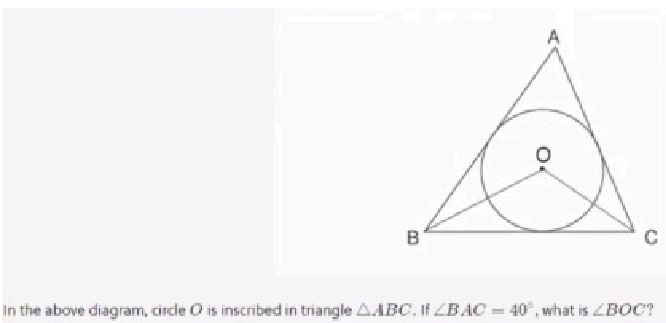
Problem 1:



Strategy: Use the properties of an inscribed circle to help solve: (primarily angle bisector property)

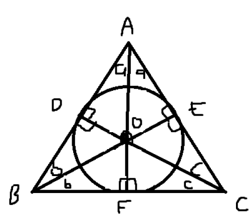
Property 1: The incenter is the intersection point of the triangle's three angle bisectors
Thus, ABO=BOC and COA=BCO and BAO=AOC;
We can now determine each of the angles in triangle ABC; 25*2,35*2, and x; x=180-120=60;
AOC=½ * ABC such ACO=30

Problem 2:



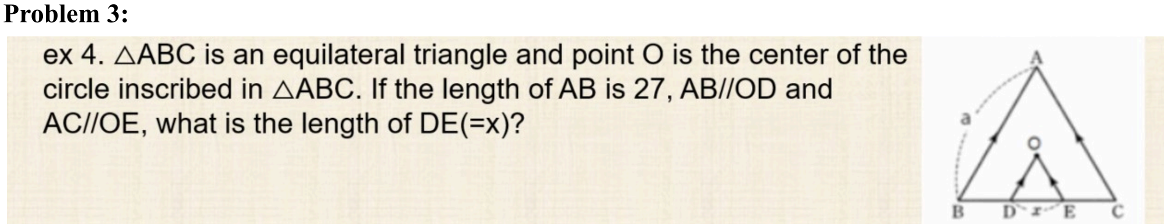
Strategy: Use the properties of inscribed circle to help solve: (both radi and bisector properties)

We can construct a more useful diagram:

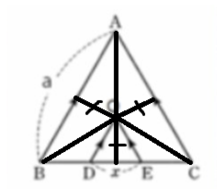


Where 2a=40;
Now we can solve this question recursively; 1. AOD=180-(90+20)=70; 2. ABC=180-(70+70)=40;
3. ABO=180-(40+90)=50; 4. B)F=180-(50+90)=40; Thus, BOC=40*2=80;

Problem 3:



Strategy: Construct a more suitable diagram and solve using similar triangles (<u>using mostly radi property</u>)



Because the larger triangle is equilateral we can divide the triangle into three congruent triangles. With this, we can determine the height of one congruent triangle to be equal to the area of triangle ABC / 3 * 27 = $\frac{\sqrt{3}}{4}27 * 27 \div 3 \div 27 = \frac{9\sqrt{3}}{4}$. We can prove that the smaller triangle, ODE, is similar to triangle ABC through its 3 parallel lines. The height of the larger triangle = 27/2*$\sqrt{3}$; The ratio between the two heights is 6:1 such that the side DE of triangle ODE = 9/2

<u>Circumscribed Circles</u>

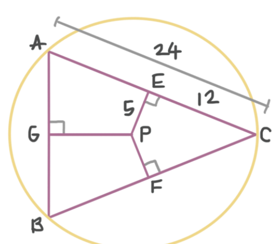
<u>Properties</u>

Main Properties:

1. The circumcenter is the point where the perpendicular bisectors of the sides of (a triangle) meet
2. The vertices of the triangle are on the circumference of the circle
3. The circumcenter is equidistant to the vertices equal to the circumscribed circle's radius (r)

Constructing circumscribed circles:

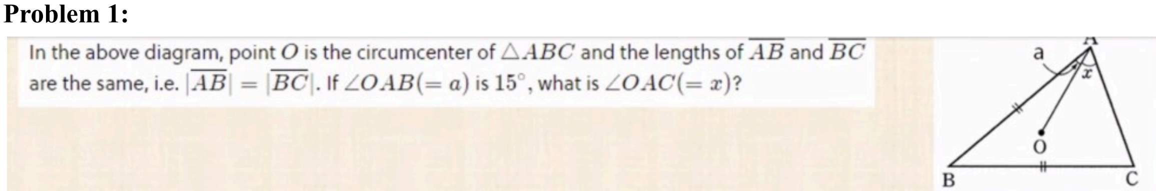
1. For an acute triangle the circumcenter is inside the triangle
2. For a right triangle the circumcenter is on the hypotenuse
3. For an obtuse triangle, the circumcenter is outside the triangle.



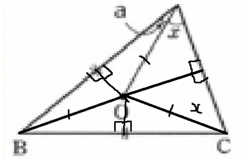
Formula: Circum radius = 4[ABC]R

<u>Problems:</u>

Problem 1:

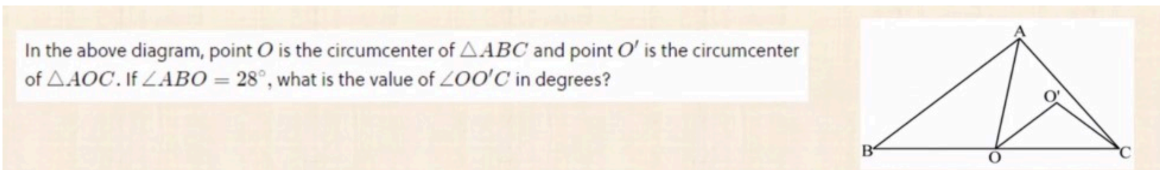

In the above diagram, point O is the circumcenter of $\triangle ABC$ and the lengths of $\overline{AB}$ and $\overline{BC}$ are the same, i.e. $|\overline{AB}| = |\overline{BC}|$. If $\angle OAB (= a)$ is 15°, what is $\angle OAC (= x)$?

Strategy: Construct a diagram that includes the properties of circumscribed triangles. Next, <u>solve using the "every line connecting a vertice to the circumcenter is equal" property.</u>

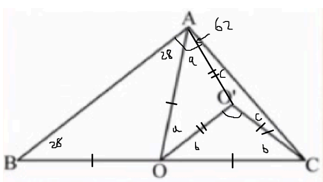
Diagram: 

With this diagram, we can see how the triangle ABO is isosceles. From this, because BAO =15, angle ABO = 15; We can also see that triangle BOC is congruent to triangle BOA given the SSS proof. From this, CBO=15 and OCB=15; We can now form the equation: 2x+15+15+15+15=180; => 2x=120; => x=60;

Problem 2:



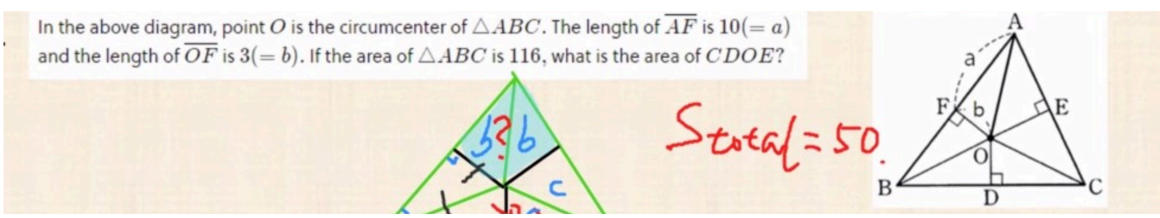
Strategy: Represent the unknown angles with variables (common when using property #3)

Diagram: 

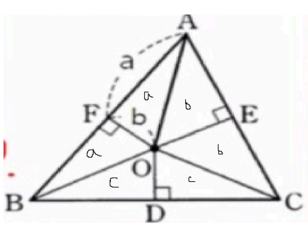
2a+2b+2c=180;

OO'C=180-2b; Thus, OO'C=2(a+c); Thus, OO'C=2*62=124;

Problem 3:



<u>Strategy:</u> Represent the different triangles present with variables (common when using property #1)



2a = 20*3 / 2 = 30;

2(a+b+c)=116;

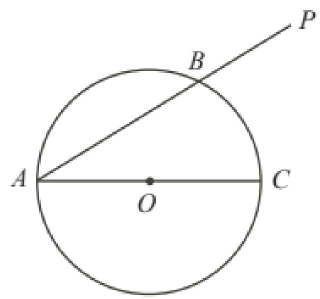
2(b+c)=116-30=86; Thus, b+c=**43**;

Proving/disproving the fact that points are concyclic

Strategies:

1. Coordinatizing the diagram
2. Contradiction is often a useful strategy when disproving concyclic properties

3. In the diagram, the circle has centre O, diameter AC, and radius 1. A chord is drawn from A to an arbitrary point B (different from A) on the circle and extended to the point P with $BP = 1$. Thus P can take many positions. Let $\mathcal{S}$ be the set of points P.



(a) Let U be a point in $\mathcal{S}$ for which UO is perpendicular to AC. Determine the length of UO.

(b) Let V be a point in $\mathcal{S}$ for which VC is perpendicular to AC. Determine the length of VC.

(c) Determine whether or not there is a circle on which all points of $\mathcal{S}$ lie.

Problem 1:

We start by coordinatizing the diagram: A=(-1,0), O=(0,0), C=(1,0);
We will attempt to employ a contradiction method to this problem: let Z be the circle that contains all points in set S;
We know that, every point above AC can be reflected about ray AC to produce another valid element within set S;
Thus, the center of Z must lie on the x-axis of our hypothetical coordinate plane.
Let the center= (p,0); The equation of z is thus: r^2=(x-p)^2+(y)^2;
From (a) we know that (0,sqrt(3)) lies on the circle Z: Thus, r^2=p^2+3;
We also know the (2,0) must lie on Z such that: r^2=(x-2)^2; Thus, the final equation of Z is:
$(x-\frac{1}{4})^2+y^2=(\frac{7}{4})^2$. From (b) we know that $\left(1,\sqrt{\frac{1+\sqrt{17}}{2}}\right)$ lies on Z;
If we plug this point in, we will see that it cannot lie on Z which is a contradiction. Thus, the points in S do not form a circle.

Trigonometry Problems:

Basic Formulas:

1. sin(x)=opposite/hypotenuse
2. cos(x)=adjacent/hypotenuse
3. tan(x)=opposite/adjacent
4. cosecant(x)=1/sin(x)
5. secant(x)=1/cos(x)
6. cotangent(x)=1/tan(x)

Identities:

Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal identities:

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Even - odd identities:

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

Co - function identities:

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x$$

Periodicity identities:

$$\sin(x \pm 2\pi) = \sin x$$

$$\cos(x \pm 2\pi) = \cos x$$

$$\tan(x \pm \pi) = \tan x$$

$$\cot(x \pm \pi) = \cot x$$

$$\sec(x \pm 2\pi) = \sec x$$

$$\csc(x \pm 2\pi) = \csc x$$

Sum and difference formulas:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Product to sum formulas:

$$\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x - y) - \cos(x + y)\right]$$

$$\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x - y) + \cos(x + y)\right]$$

$$\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x + y) + \sin(x - y)\right]$$

$$\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x + y) - \sin(x - y)\right]$$

Half - angle formulas:

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{(1 - \cos x)}{\sin x}$$

Sum to product:

$$\sin x \pm \sin y = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

Law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

Double - angle formulas:

$$\sin 2\theta = 2 \cdot \sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Area of triangle:

$$\frac{1}{2}ab\sin C$$

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$

Common Angles:

We can use special triangles to determine the trigonometric ratios of their angles. For example, we know that sin(45)=1/sqrt(2) given that 45 is part of a 45-45-90 triangle that has angles in the ratio 1-1-sqrt(2).

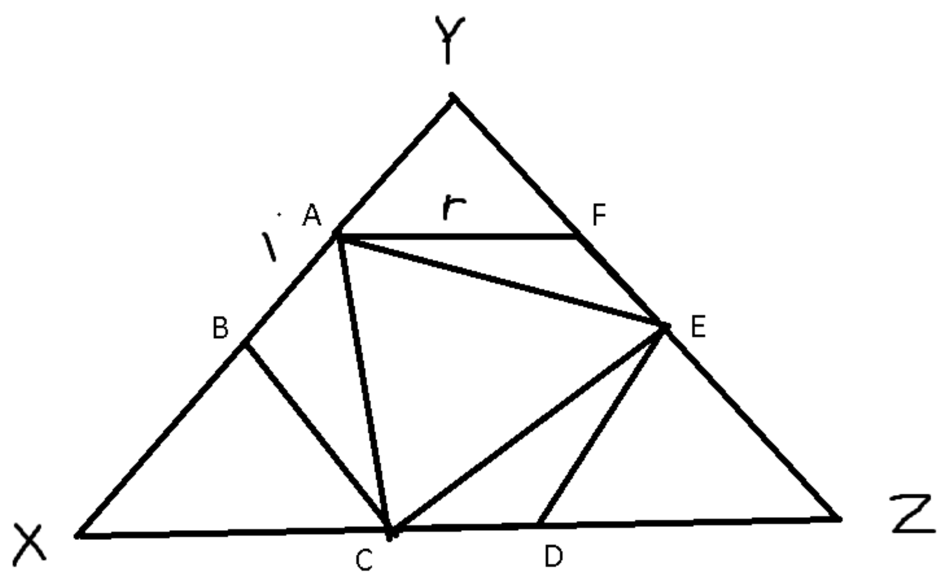
Problems:

Problem 19

Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

(A) $\frac{4\sqrt{3}}{3}$ **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{17}{4}$ **(E)** 6

Solution



By cosine law, AC = r^2+r+1; Thus, Sace = sqrt(3)/4 * (r^2+r+1);
Now we draw X,Y,Z such that XYZ is an equilateral triangle.
Thus, [(r+2)^2-3]*sqrt(3)/4
Now we can solve by making an effective equation

Miscellaneous Geometry Problems:

Problem Types:

1. Number of unit squares a hypotenuse line passes through

Number of unit squares a hypotenuse line passes through:

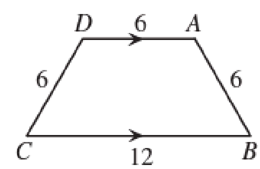
Problem 1: A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

Formula: l+w-gcd(l,w)

Thus, answer is 10+17-1=26

<u>"Leash" problems:</u>

2. A barn has a foundation in the shape of a trapezoid, with three sides of length 6 m, and one side of length 12 m, as shown.



(a) Determine each of the interior angles in the trapezoid.

(b) Chuck the Llama is attached by a chain to a point on the outside wall of the barn. Chuck is smarter than the average llama, and so realizes that he can always reach the area between the barn and where the chain is fully extended.

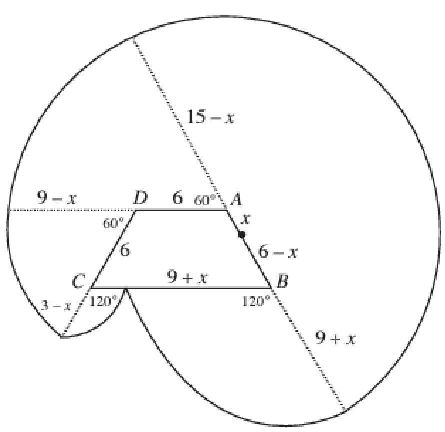
(i) If Chuck is attached at the point A with a chain of length 8 m, what is the area outside the barn that Chuck can reach?

(ii) If Chuck is attached at some point P along the wall between A and B with a chain of length 15 m, determine the location of P which restricts Chuck to the *minimum* area.

Problem 1:

(b):

(ii): We assume a point x units from point A on AB. Such that:



It's important to note that when 0<=x<=3 the area equation may differ from when 3<=x<=6; this is because the "leash" may wrap around C in the latter case, which may affect the area.

Solid Geometry:

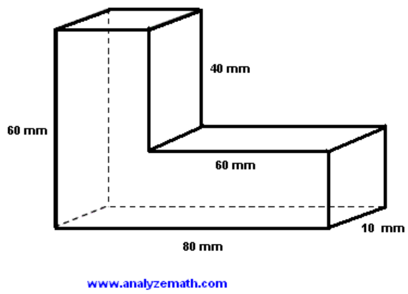
Types of Problems:

1. Basic Implementation
2. Creating Cross-sections

3. Using similarity
4. Water Problems
5. Creating Diagrams

Basic Implementation:

Problem 1:



Find the volume of the given L-shaped rectangular structure.

Target: Total Volume:
Strategy: Split into sections (If not already calculable) and solve.
Section 1: 60*10*20=12000;
Section 2: 60*20*10=12000;
Total = 24000;

Creating Cross-sections:

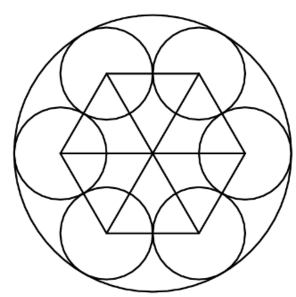
Problem 1:

Problem 22

Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

(A) $\sqrt{2}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** $\sqrt{3}$ **(E)** 2

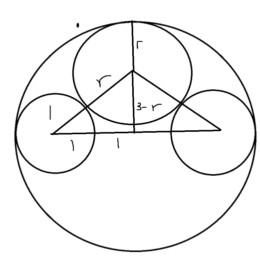
When dealing with complex solid geometry problems, the solution almost always involves converting 3d to 2d. We do this by creating cross sections at symmetrical axis:



1. We start by looking at the diagram from **birds-eye-view:**

Here, it is clear that the radius of the large sphere must equal 3 (due to the hexagon's equilateral triangles)

2. Next lets look at it from the side:



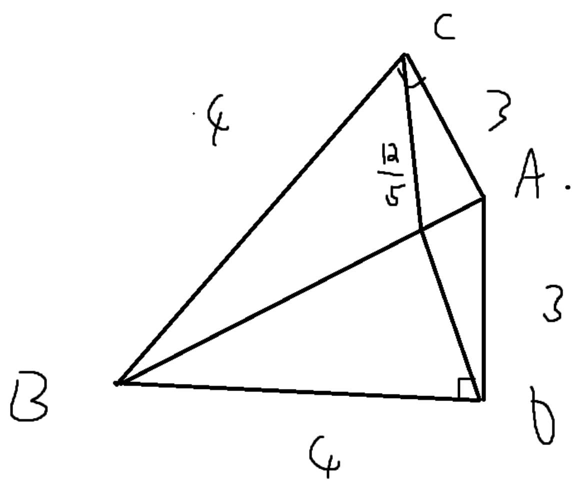
If we assume the radius of the 8th circle to be r, then we can solve for it by creating a right triangle and using the facts we gained from the **birds-eye-view**

Problem 2:

Tetrahedron $ABCD$ has $AB = 5$, $AC = 3$, $BC = 4$, $BD = 4$, $AD = 3$, and $CD = \frac{12}{5}\sqrt{2}$. What is the volume of the tetrahedron?

(A) $3\sqrt{2}$ **(B)** $2\sqrt{5}$ **(C)** $\frac{24}{5}$ **(D)** $3\sqrt{3}$ **(E)** $\frac{24}{5}\sqrt{2}$

This problem requires a deep understanding of 2d-heights and their properties in solid geometry:

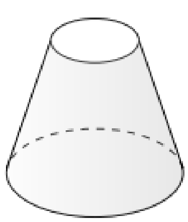


If we draw the diagram, we recognize that the height relative to the base triangle ABD must be 12/5 as it forms a 90 degree angle about base AB and a 90 degree angle about the perpendicular line from D to AB.

We know the second statement is true as the triangle formed by connecting CD is a 45-45-90 triangle as their sides are in ratio 1:1:sqrt(2)

<u>Using Similarity:</u>

Problem 1: A cone of height 6cm and radius 6cm has the top cut out. The resulting solid is called a truncated cone, which is shown in the diagram below. If the height of this truncated cone is 3cm, what is its volume? (Use pi = 22/7 as an approximation

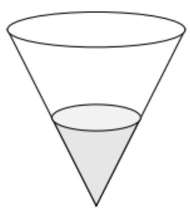


Target: Volume of figure given;
Strategy: Determine the volume of the unembodied section of the cone through similarity and then determine the difference between the total and unembodied section.

Total Volume = 72π;
Unembodied height : total height = 1:2;
Volume ratio = 1:8;
$\frac{1}{8}$*72*pi=9 π;
Truncated cone volume = (72-9) π = 63 * 22/7 = 198;

Water Problems

Problem 1: A cup, in the shape of an inverted right circular cone, has height 10 cm and base radius 5 cm. The water level is such that it takes 2.7% of the volume of the cup. What is the distance from the top of the water to the top of the cup ?



Target: Difference between the large height and the height of the water;
Strategy: Find the ratio between the volumes of the water to find the ratio between the small and large heights; Then, using the larger height, solve for the smaller height and determine the difference.

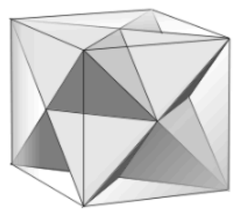
Area ratio: 2.7: 100 => 27:1000; Side length ratio = cube root of volume ratio;
Side ratio = 3:10;
Height 1 = 10; Height 2 = 3; Difference = 7;

Creating Diagrams

Problem 1:

> Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra?
>
> (A) $\frac{1}{12}$ (B) $\frac{\sqrt{2}}{12}$ (C) $\frac{\sqrt{3}}{12}$ (D) $\frac{1}{6}$ (E) $\frac{\sqrt{2}}{6}$

Solution:



A regular unit tetrahedron can be split into eight tetrahedra that have lengths of $\frac{1}{2}$. The volume of a regular tetrahedron can be found using base area and height:

For a tetrahedron of side length 1, its base area is $\frac{\sqrt{3}}{4}$, and its height can be found using Pythagoras' Theorem. Its height is $\sqrt{1^2 - \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}$. Its volume is $\frac{1}{3} \times \frac{\sqrt{3}}{4} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{12}$.

The tetrahedron actually has side length $\sqrt{2}$, so the actual volume is $\frac{\sqrt{2}}{12} \times \sqrt{2}^3 = \frac{1}{3}$.

On the eight small tetrahedra, the four tetrahedra on the corners of the large tetrahedra are not inside the other large tetrahedra. Thus, $\frac{4}{8} = \frac{1}{2}$ of the large tetrahedra will not be inside the other large tetrahedra.

The intersection of the two tetrahedra is thus $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = \boxed{\text{(D)}}$.

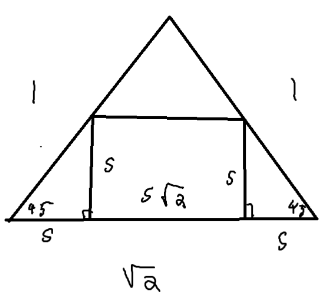
Problem 2:

A pyramid has a square base with sides of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube?

(A) $5\sqrt{2} - 7$ **(B)** $7 - 4\sqrt{3}$ **(C)** $\frac{2\sqrt{2}}{27}$ **(D)** $\frac{\sqrt{2}}{9}$ **(E)** $\frac{\sqrt{3}}{9}$

Strategy: Create a diagram of the described solid object and convert into a 2d shape.

Here, the best 2d shape to use is the side profile of the pyramid:



Now we can solve for s.

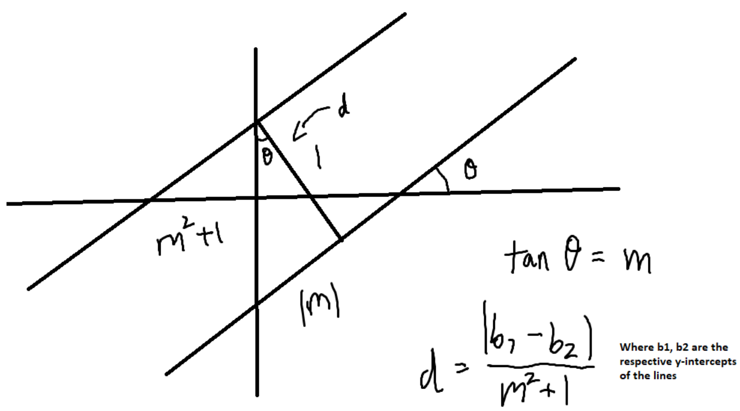
Analytic Geometry:

<u>Coordinate Plane Basics:</u>

Formulas:

1. Slope = rise / run = (y1-y1)/(x1-x2);

2. Midpoint = ((x1+x2)/2), ((y1+y2)/2);
3. Distance = $d=\sqrt{((x2-x1)^2+(y2-y1)^2)}$;
4. If a two lines are perpendicular: m1 * m2 = -1 (where m1, m2 are the respective slopes of the lines).
5. Intersection point between two lines is equal to the solution of there two equations
6. Slope-intercept fórm of a straight line: y=mx+b; (where b = the y-intercept); *The equation of a vertical line does not have a y-intercept since a vertical line never crosses the y-axis.*
7. General Form of a straight line: Ax + By + C =0; where A,B, C are constants and A,B are not both 0;
8. Perpendicular Lines in General Form: a1/b1*a2/b2=-1 (where the equations of the two line are a1x+b1y+c1=0 and a2x+b2y+c2=0)
9. Parallel Lines in General Form: a1/a2=b1/b2!=c1/c2 (where the equations of the two line are a1x+b1y+c1=0 and a2x+b2y+c2=0)
10. Converting from general to slope-intercept:
    We have the general equation Ax + By + C = 0.
    If $b \neq 0$, then from the given equation we get,
    y = (-A/B)x+(-C/B);
11. Solving for the distance between two parallel lines:
    |b1-b2|/sqrt(1+m^2) = d (Where the two lines are in the form y=mx+b1, y=mx+b2)
    |c1-c2|/sqrt(a^2+b^2) = d (Where the two lines are in the form ax+by+c1=0, ax+by+c2=0)
    Proof:



Typical Types of Problems:
1. Basic Implementation of formulas
2. Solving for area in coordinate plane
3. Reflection, translation, and rotation
    - Reflection
    - Translation
    - Rotation

<u>Basic Implementation of Formulas:</u>

Parallel lines: (Same Slope)

Problem 1: Line L is represented by the equation 2x−3y=4. If line m passes through the points (1,4) and (2,a), and if m is parallel to l, then what is the value of a?

Target: The value of a;
Strategy: Determine the slope of line L (because it is the same as that of line m); Form an equation and substitute the first pair of coordinates. Once the y-intercept is solved, substitute the second pair and solve for a;

Slope of L: y=2/3x-4/3; Slope = 2/3;
Equation of m: y=2/3x + b; => 4 = ⅔ + b; => b=3⅓; y=2/3x + 10/3;
a=4/3 + 10/3 = 14/3;

Problem 2:

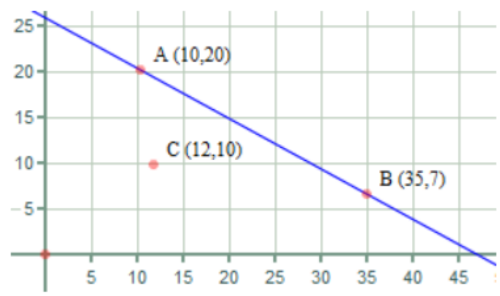


FIG 2. WE NEED A LINE PARALLEL TO AB THROUGH C

Target: The equation of the line which passes through C;
Strategy: Solve for the slope of the line passing through A and B. Then, create an equation for the target line in slope-intercept form. Lastly, substitute the coordinates of C into the equation to determine the equations y-intercept.

AB slope = 20-7/10-35=13/-25;
Target line equation: y=13/25 * x + b;
Substitute C: 10=13/25*12+b; => b=94/25;

Perpendicular Lines:

Problem 3: What line is perpendicular to $x + 3y = 6$ and travels through point (1,5)?

Target: Equation of line;
Strategy: Determine the slope of the first line and use the perpendicular line property to solve for the slope of the target line; After, substitute the coordinates and solve for the equation.

Slope of first line: 3y=6-x; => y=2-1/3x; slope = -⅓;
Slope of target line = 3;
Equation: y=3x + b; => 5=3+b; b=2;
Final Equation: y=3x + 2;
Problem 4: Determine the equation in the general form of the line perpendicular to 2x + 5y + 10 =0 with the same x-intercept as 3x - 2y = 12?

Target: Equation of a line;
Strategy: Convert the general form to slope-intercept and solve for slope; Create an equation with this new data, representing the target line. Solve for the x-intercept of the second equation. Substitute (x-intercept,0) into the target equation;

2x+5y+10=0; => y=-2/5x-2; Slope = $-\frac{2}{5}$;
3x-2y=12; => 2y=3x-12; y=3/2x-6; let y=0; x=4;
Target slope = 5/2;
Equation: y=5/2x+b; Sub (4,0); b=-10;
Equation: y=5/2x-10;

Midpoint + Distance:

Problem 5: Find the distance between the two points with coordinates given as, A = (1, 2) and B = (1, 5).

Target: Distance between the points;
Strategy: Use formula to solve;
Formula: $d=\sqrt{(x2-x1)^2+(y2-y1)^2}$;
$d=\sqrt{(x2-x1)^2+(y2-y1)^2}$; => sqrt(0^2+3^2)=3 units;

Problem 6: Line segment AB has an endpoint, A, located at (10,−1), and a midpoint at (10,0). What are the coordinates for point B of segment AB?

Target: Coordinates of point B;
Strategy: Use the midpoint formula to set up an equation and solve;
Formula: ((x1+x2)/2),((y1+y2)/2); => (10+x2)/2, (-1 + y2)/2 = 10, 0;
x2=10; y2=1;
Coordinates = 10,1;

Problem 7:

Problem 20

A dilation of the plane—that is, a size transformation with a positive scale factor—sends the circle of radius 2 centered at $A(2,2)$ to the circle of radius 3 centered at $A'(5,6)$. What distance does the origin $O(0,0)$, move under this transformation?

(A) 0 **(B)** 3 **(C)** $\sqrt{13}$ **(D)** 4 **(E)** 5

<u>Solving for Area:</u>

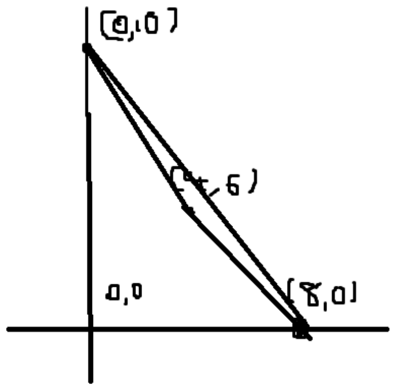
Strategy for most questions: If not given, create a diagram (doesn't have to be exact). Then, split the shape into multiple valuable shapes and solve in sections.

Problem 1: What is the area of the triangle formed by the two lines y=1/2x+4,y=4x-10 and 5x + 4y + 40 =0?

Target: Area of the shape given on the plane;
Strategy: Find the intersection points of the lines, create a diagram, and split into calculable shapes.

5x+4y+40=0; => y=-5/4x-10;
Intersection 1: y=1/2x + 4; y=4x-10; => 8y=4x+32; y=4x-10; => 7y=42; y=6 x=4; (4,6)
Intersection 2: y=1/2x+4; y=-5/4x-10; => 1/2x+4=-5/4x-10; => 7/4x=-14; x=-8,y=0; (8,0);

Intersection 3: y=4x-10; y=-5/4x-10; => -5/4x=4x; => 21/4x=0; x=0 y=-10; (0,10);



Midpoint = (0+8)/2,(10+0)/2=4,5;
Section 1 = The section from the midpoint to the highest point;
Section 2 = The section from the midpoint to the lowest point;

Section 1 = 1*4/2=2;
Section 2 = 1*4/2=2;
Total Area = 4 units;

<u>Reflection and Rotation:</u>

<u>Reflection</u>

Reflecting Singular lines:

<u>How to find image point (Given one point and line of reflection):</u>

Step 1: Assume the image point and give variable coordinates (a,b);
Step 2: Determine the equation of the line connecting both the points (perpendicular to line of reflection)
Step 3: Use midpoint formula to set up two equations in terms of a and b;
Step 4: Solve equation;

<u>How to find line of reflection equation (Given both points):</u>

Step 1: Assume the line of symmetry;
Step 2: Determine the equation of the line connecting both reflected points;
Step 3: Create an equation for the reflection line, (perpendicular to connection line), with unknown y-intercept
Step 4: Substitute the midpoint of the two points into the equation and solve for the y-intercept;

Formulas:

1. A reflection through the line, y=x: (x,y)=(y,x)

Proof:

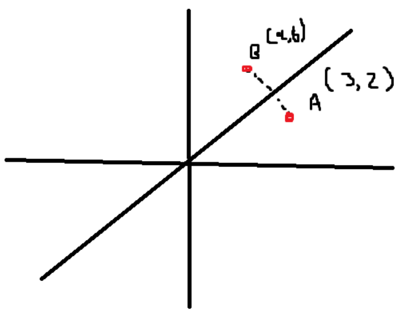
2. A reflection through the line, y=-x: (x,y)=(-y,-x)

Problem 1: You are given a point A of coordinates (3,2). A line of reflection with equation y=x reflecting point A; Find the coordinates of point A's reflection;

Target: Equation of line;

Strategy: Assume reflected point; Determine the equation of the line connecting the reflected point; Determine the midpoint of the reflected points in terms of the target coordinates; Make a system of equations with both the connecting line equation and the midpoint equation and solve.

Let point A's reflection = B;

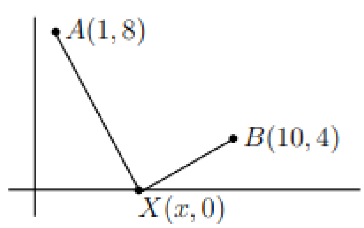


Reflecting line slope = 1;
AB slope = -1;
AB equation: y=-1x+b; => 2=3+b; => b=-1;
Midpoint equation: (b+2)/2=(a+3)/2;
System of equations: b=a-1; (b+2)/2=(a+3)/2; a=2; b=3
Coordinates = (2,3);

<u>Minimum Distance Problems:</u>

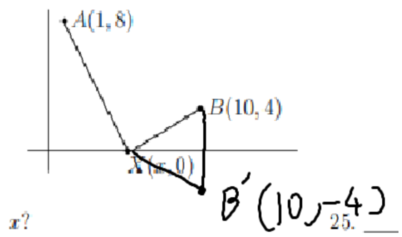
Problem 1:

24. Points $A(1,8)$ and $B(10,4)$ have been connected by lines to point $X(x,0)$. What is the smallest possible value of $AX + XB$? 24. ____________



25. In Question 24, what is the value of x? 25. ____________

24) The optimal position for x such that the length of AX+XB is minimized is not obvious given the current position of points A and B. A solution to this issue, is to reflect point B about the x-axis to form a point B' (B' is always the same distance to X as B).



It can now be observed that the optimal point X such that AX + XB' is minimized will be when X is on line AB'. Thus, the minimal value of AX + XB' is just the value of AB' = sqrt((10-1)^2+(8-(-4))^2) = 15.

25) We can solve for the value of point X by solving for the equation of line AB' and substituting y=0 into the equation.

Slope of AB' m(ab')=-12/9=-4/3;
AB': y=-4/3x+b (substiute point A)
8=-4/3+b
b=28/3
y=-4/3x+28/3
Let y=0;
x=7

<u>3d-reflection</u>:

Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m + n$.

The logic behind this question is that each vertex of the cube has coordinates (x,y,z) where x,y,z = (12,0) each
Thus, the total distance travelled from one vertex to another (no matter the route) must be divisible by 12:
Because 5 and 7 (the x and y distances travelled) are relatively prime to 12, the light must reflect at least 12 times.

We know that the distance travelled in one reflection is $\sqrt{(12^2) + (5^2) + (7^2)} = \sqrt{218}$.

Thus, 12 reflections is $12\sqrt{218}$. units travelled

<u>Rotation:</u>
Formulas:

1. A rotation of 90 degrees counterclockwise about the origin is equivalent to the coordinate transformation

(*x*,*y*)→(−*y*,*x*).

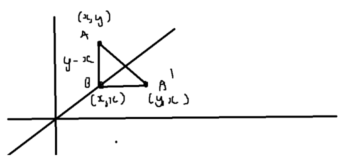
2. A rotation of 180 degrees counterclockwise about the origin is equivalent to the coordinate transformation (*x*,*y*)→(−*x*,−*y*).

3. A rotation of 270 degrees counterclockwise about the origin is equivalent to the coordinate transformation (*x*,*y*)→(*y*,−*x*).

4. A rotation of 360 degrees about the origin is equivalent to a rotation of 0 degrees and both are equivalent to the coordinate transformation (*x*,*y*)→(*x*,*y*).

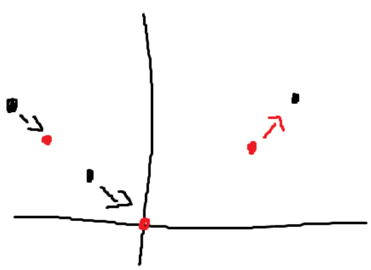
5. Line segments and shapes can be rotated by applying coordinate transformations to each of their endpoints or vertices.

Proofs for reflecting about y=x and y=-x:



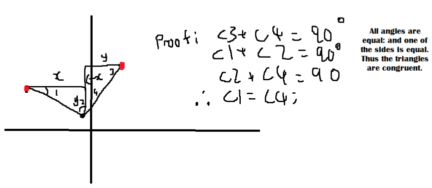
Steps: (when rotation point is not the origin) :

Method 1: Translate the rotation point to the origin and translate the point being rotated by the same amount; After, use the formulas provided to rotate the point by the degrees requested; Finally, move the image point by the same amount that it was moved in the first step, but in the opposite directions.
Diagram:



Method 2: Make two congruent triangles to solve:

Diagram:



<u>Dilation:</u>

When dilating a shape, you are essentially creating a shape that is similar, and whose sides are in the same ratio as the scale factor. Two common methods for dilating a shape are 1. Transforming the points such that the point of dilation is located at the origin, using the dilation formula, and transforming the points back, essentially erasing the transformation at the start. 2. Using plane geometry and similar triangles.

Dilation formula:

When the point of dilation is located at the origin, then, if the original point has coordinates (x,y) and the scale factor is z, then the image point has coordinates (z*x,z*y);

Dilation Problems:

Problem 1:

27.__________ A segment with endpoints $G(-2, 3)$ and $H(4, 7)$ is dilated by a scale factor of $\frac{2}{3}$ with center of dilation (0, 0). What is the sum of all the coordinates of G' and H'?

Solution (Using the dilation formula):

G'=(-2*2/3,3*2/3)=(-4/3,2);
H'=(4*2/3,7*2/3)=(8/3,14/3);
Sum = 18/3+2=8;

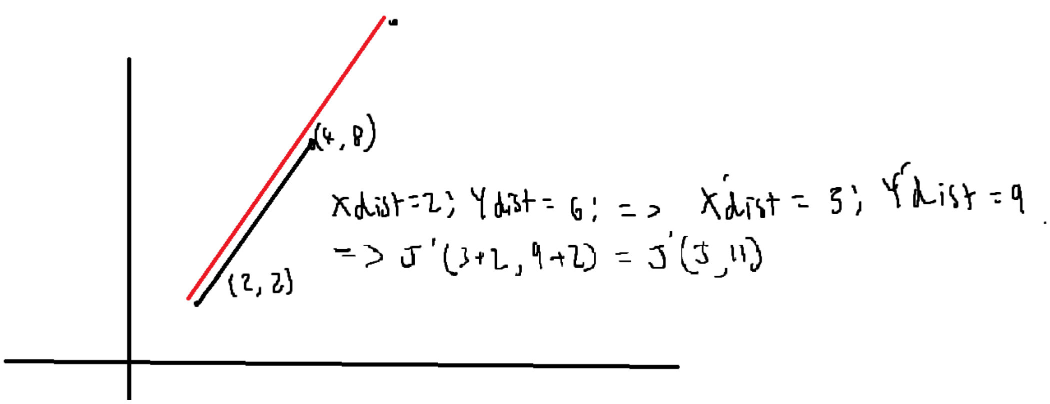
Problem 2:

28.__________ Point $J(4, 8)$ is dilated by a scale factor of $\frac{3}{2}$ with center of dilation $K(2, 2)$. What is the product of the coordinates of J'?

Method 1: (Transformation Method)

J(4,8)=>J'(2,6); K(2,2)=>K(0,0);
J''=(2*3/2,6*3/2)=(3,9)=>J'''=(5,11)
Product= 55;

Method 2: (Using plane geometry):

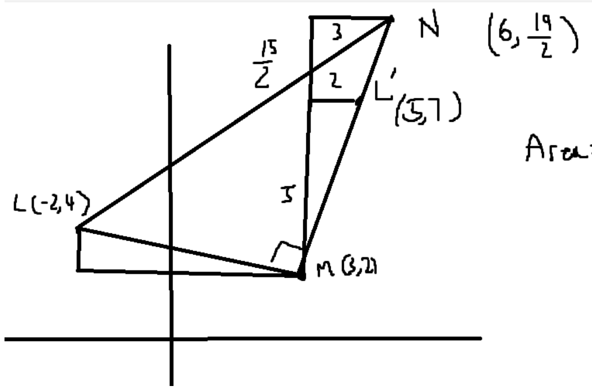


Problem 3:

29.______ units² A point L(−2, 4) is rotated 90 degrees clockwise about the point M(3, 2). Point N is the image of L′ dilated by a scale factor of $\frac{3}{2}$ with center of dilation M. What is the area of $\triangle$LMN? Express your answer as a common fraction.

Method: Rotate the point L 90 degrees about point M(3,2). 2. Dilate the image point by a scale factor of 3/2 about the point M(3,2);

We can use either method, but for my convenience, I will just use the plane geometry method.



$$\text{Area} = \sqrt{29} \times \frac{3}{2}\sqrt{29} = \frac{3 \cdot 29}{2} = \frac{87}{2} \times \frac{1}{2} = \frac{87}{4}$$

An important aspect of this problem is that the calculation of coordinates (L' and N) was not actually required. We can simply calculate the distance between L and M and then subsequently multiply that distance by the scale factor, 3/2. These two results will be the two shorter legs in a right triangle and can be multiplied and divided by two to produce the area.

Lattic Point Rotation:

Problem 1:

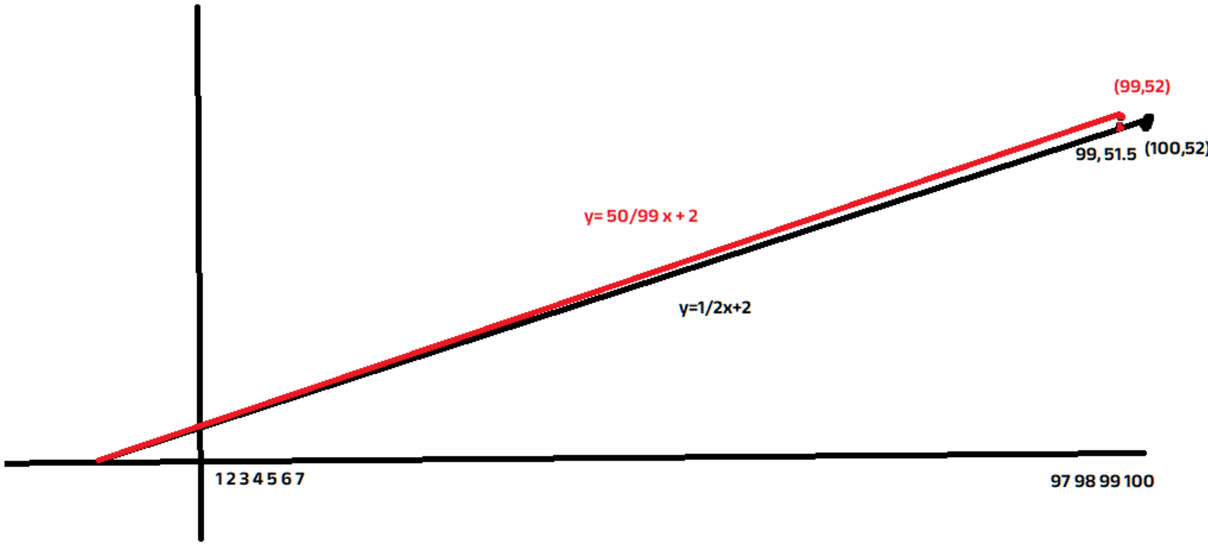
Problem 24

A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $1/2 < m < a$. What is the maximum possible value of a?

$\textbf{(A)}\ \frac{51}{101} \qquad \textbf{(B)}\ \frac{50}{99} \qquad \textbf{(C)}\ \frac{51}{100} \qquad \textbf{(D)}\ \frac{52}{101} \qquad \textbf{(E)}\ \frac{13}{25}$

Solution

In these problems, the best approach is to graph a line with minimum value slope m, and then check the end lattice points. Then, intuitively figure out the "closest" lattice point and determine the slope of the line that connects between that lattic point and the y-intercept:



Once this is done, the answer becomes obvious: y= 50/99x+2; Thus, a=**55/99**

Number Theory:

<u>Diophantine Equations:</u>

A diophantine equation is an equation that has more than one variable, integer coefficients, and integer solutions (eg. 4x+5y=8). There are many different forms in which diophantine equations are commonly presented. I will summarize the strategies for a few:

1. <u>Simon's Favorite Factoring Trick:</u>

   a. **Problem 1: (Easy)**

      A rectangular floor measures a by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

      $\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

      The area of the painted rectangle is (a-2)(b-2). This value equals ½(ab) such that:

      ab-2a-2b+4=1/2ab

      (a-4)(b-4)=8

      a,b=(6,8),(8,6),(5,12),(12,5)

   b. **Problem 2: (Medium)**

1. The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that N is a perfect square.

(y-N)(x-N)=N^2. Thus, d(N^2)=2005. Thus, N^2=p1^2004 or N^2=p1^4*p2^400

Both of these possibilities result in N being a perfect square. Thus, N most be a perfect square if the conditions are met.

2. <u>Linear Diphatine Equations (Euclidean Algorithm + Bezout's Identity)</u>

a. **Problem 1: (Easy)**

$$57x + 22y = 400$$

One strategy to solve problems in this linear function form is to use the Eucladian Algorithm in reverse:

1. Find the GCD(57,22)=1. [if GCD(a,b)|c then there are integer solutions to the diphatine equation]
2. Solve for the GCD using the algorithm rather than simple inspection:
   a. GCD=(57-22*2,22-[57-22*2])

      GCD=(57-22*2-[22*3-57],22*3-57)

      GCD=(2*57-22*5,22*3-57-[4*57-10*22])

      1=13*22-5*57

      400=5200*22-2000*57

      Thus, one of the infinite possible solutions for pair (x,y)=(-2000,5200)

      To develop a formula, x=-2000+22/(GCD[52,22])*t=**-2000+22t**;

b. **Problem 2: (Medium)**

Find the smallest integer k for which the conditions

(1) $a_1, a_2, a_3, \ldots$ is a nondecreasing sequence of positive integers

(2) $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$

(3) $a_9 = k$

are satisfied by more than one sequence.

First we map out the sequenceto the 9th term. From this, we get that a_9=21a+13b (when a,b=a_1,a_2)

We know that k=21b+13a (for integers a,b,x,y)

Because gcd(21,13)=1, then a_1 = a+21, a_2=b-13; (where x,y are the smallest integers greater than a,b). We know that a_2>=a_1; b-13>=a+21, b>=35

Thus, a_1,a_2=(1,35) or (22,22)

3. Assuming relationship between variables:

Problem 1: In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that x < y < z are constructed. How many such triangles are there?

Strategy: Create expressions from the conditions given, manipulate them to make them more usable, and use the expressions to count effectively.

x+y+z=57; x<y<z; 3x<x+y+z<3z; => 3x<57<3z<57/2; => x<19<z<29; Now we can use these restrictions to count the different cases.

If z=28; x+y=29;
x,y=2,27; x,y=14,15; Total=13 possibilities
If z=27; x+y=30;
x,y=4,26; x,y=14,16; Total=11 possibilities
If z=26; x+y=31;
x,y=6,25; x,y=15,16; Total=10 possabilities
If z=25; x+y=32;
x,y=8,24; x+y=15,17; Total 8 possabilities;
If z=24; x+y=33;
x,y=10,23; x,y=16,17; Total 7 possibilities;
If z=23; x+y=34;
…
If z=21; x+y=36;
x,y=16,20; x,y=17,19; Total = 2 possibilities
If z=20; x+y=37;
x,y=18,19; Total = 1 possibility such that the total sum of possibilities = 61

Problem 2:

30. ______________ If *a*, *b* and *c* are positive integers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{6}{7}$, then what is $a + b + c$?

Assume a<=b<=c; Thus, 1/a>=1/b>=1/c;
Thus, 3/a>=6/7; => a<=7/2 => a<=3;
There are two plausible values for a, 2 and 3, given that, if a were equal to 1, then the sum of 1/b and 1/c would be negative.

Let a=3; Thus, 1/b+1/c=6/7-1/3=11/21; => b<=42/11; => b<=3; b>a, thus b=3; If this is true, 1/c=6/7-2/3 ; => 4/21; Which is invalid.

Let a = 2; Thus, 1/b+1/c=6/7-1/2=5/14; Thus, 2/b>=5/14; => b<=28/5; => b<=5;
b cannot equal because that would result in c being negative.
If b = 3: 1/c=6/7-1/2-1/3; => 1/c=1/42; This solution is valid. Thus, a=2,b=3.c=42; Sum=47;

The positive integers a, b, and c satisfy the equation

$$\frac{4}{5} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

What is the largest possible value of $a + b + c$?

Problem 3:

Assume a<=b<=c; Thus, 1/a>=1/b>=1/c; 3/a>=4/5; => a<=15/4; => a<=3;

a cannot equal 1, because then at least one of the other variables would need to be negative. Thus, a=2 or 3;

We can assume a=2; 4/5=½+1/b+1/c; => 3/10=1/b+1/c; 2/b>=3/10; b<=20/3; b<=6; b could equal 6,5,4,3,2; For c to be positive, b must be >=4; So the actual possible values are 4,5,6;

So, we can systematically assume b is equal to each of these values and see if c is an integer;
b=4; 1/c+1/b=3/10; 6/20-5/20=1/20; When b is 4, c=20; One possible triple is (2,4,20);
b=5; 1/c+1/b=3/10; 3/10–2/10=1/c; When b is 5, c=10; Another triple is (2,5,10);
b=6; 1/c=9/30-5/30=4/30; This results in c not being an integer and is thus an invalid value;

Now we assume a=3; 1/3+1/b+1/c=4/5;

1/b+1/c=12/15-5/15=7/15; Because 1/b>=1/c; 2/b>=7/15; b<=30/7; b<=4; b cannot be less than 3 because c would then be negative;

b=3; 1/c=7/15-5/15=2/15; If b=3; c is not an integer;
b=4; 1/c=28/60-15/60=13/60; c is not an integer;

Thus the greatest triple is (2,4,20);
So 26 is the greatest sum;

4. Divisibility:

8. Determine the number of triples (k, l, m) of positive integers such that

$$k + l + m = 97$$

$$\frac{4k}{5} + \frac{5l}{6} + \frac{6m}{7} = 82$$

Problem 1:

We know that 5|k, 6|l, 7|m. Thus, we can assume k=5a, l= 6b, m=7c. Substituting these values:

5a+6b+7c=97;

4a+5b+6c=82

=> a+b+c=15

=> b+2c=22;

Thus, (a,b,c) = **(3,2,10), (2,4,9), (1,6,8)**

<u>Modular Arithmetic:</u>

<u>Typical Types of Problems:</u>

1. Combining Modulo Statements
2. Reducing Repeating Groups
3. Implementing the Properties
4. Deducing Patterns
5. Using Algebra + Modular Arithmetic

<u>Basic Formulas and properties:</u>

1. Addition Rules
    - If a+b=c, then a (mod m) + b(mod m) = c(mod m);
    - If a ≡ b(mod m), then a+k ≡ b + k(mod m) for any integer
    - If a≡b(mod m) and c≡d (mod m), we have a+c≡b+d(mod m)
    - If a≡b(mod m), then -a ≡ -b(mod m)
2. Multiplication Rules
    - a*b=c, then a(mod m) * b(mod m) ≡c(mod m);
    - If a≡ b(mod m), then ka ≡ kb (mod m) for any integer k;
    - If a ≡ b (mod m) and c≡d (mod m), we have a*c=b*d(mod m);
3. Properties and Strategies:
    - **Alternate remainder**: Residues can also be represented with negative values such that each modular equation contains two solutions. If N mod a = c; then, N mod a = c-a
    - **Chinese Remainder Theorem**: If N mod a = c; and N mod b = c; then N=LCM(a,b)k+c
        - Also, when a,b are relatively prime, there exists a distinct solution for N modulo ab

Problem 1: (Chinese Remainder Theorem)

Example 2.2.2 (AIME II 2012). *For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is* 3, 4, 5, 6, 7, 13, 14, 15, 16, 17 *Find the number of positive integers less than or equal to* 10,000 *which are simultaneously 7-safe, 11-safe, and 13-safe.*

If x is 7-safe, 11-safe, and 13-safe:

x ≡ 3,4 (mod 7)
x ≡ 3,4,5,6,7,8 (mod 11)
x ≡ 3,4,5,6,7,8,9,10 (mod 13)

We know that, by Chinese Remainder Theorem, these three congruences will yield 1 unique value for x from 1 to 7*11*13=1001. Thus, there are 2*6*8=96 "safe" values within that range.

(1->1001), (1002->2002), … (9009->10009)

These are 10 groups all of which have 96 "safe" values. Thus, there are 96*10=960 valid values <=10009. Notice that the question specifies for values <=10000.

10001 is invalid (remainder 5)
10002 is invalid (6)

10003 is invalid (0)
10004 is invalid (1)
10005 is invalid (2)
10006 is valid
10007 is valid

Thus answer = 960-2=958.

Problem 2: (Divisilibty rules + pigeon hole principle)

Example 2.2.3. *Consider a number line consisting of all positive integers greater than 7. A hole punch traverses the number line, starting frrom 7 and working its way up. It checks each positive integer n and punches it if and only if $\binom{n}{7}$ is divisible by 12. (Here $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.) As the hole punch checks more and more numbers, the fraction of checked numbers that are punched approaches a limiting number ρ. If ρ can be written in the form $\frac{m}{n}$, where m and n are positive integers, find $m + n$.*

7Cn = n(n−1)(n−2)(n−3)(n−4)(n−5)(n−6) / (2^4·3^2·5·7); By pigeon hole theorem, this number must be divisible by 7 and 5, so we must find the values for n such that 2^6*3^3 | n(n−1)(n−2)(n−3)(n−4)(n−5)(n−6);

We start by finding valid n's for 3^3: If n ≡ 0 (mod 3). The number must be divisible by 3^3. If n ≡ 1,2 (mod 3), the number is divisible by 3^2. For the term to be divisible by 3^3, one of the factors must be divisible by 3^2 (instead of just 3). This is true if n ≡ 0,1,2,3,4,5,6. (mod 9)

Next we can find valid n's for 2^6: If n is even, then 4 of the factors must be divisible by 2 (and 2 of those 4 are also divisible by 4). Thus, if n is even, the term is definitely divisible by 2^6.

If n is odd either:

1. (n-3) is divisble 4 (in which case it must be divisible by 16)
    a. n ≡ 3 (mod 16)
2. (n-1) and (n-5) are divisible by 4 (in which case one of the two must be divisible by 8)
    a. n ≡ 1, 5 (mod 8) or n ≡ 1,5,9,13 (mod 16)

Thus, n ≡ 0,1,2,3,4,5,6,8,9,10,12,13,14 (mod 16) and n ≡ 0,1,2,3,4,5,6 (mod 9)

Thus, the probability that the sequence of values is approaching is (13*7)/(9*16)=91/144

<u>Properties of Numbers:</u>

<u>Content:</u>

1. Evens and Odds
    - Properties (Addition, Multiplication, etc.)
    - Typical Problems
2. Primes
    - Common Occurrences
    - Largest Prime Divisor Problems
    - Prime Factorization Problems

3. Factors, and Multiples
    - Implementing Factors
    - Counting Divisors (with or without restraints)
    - Manipulation based on divisibility
    - Factor counting
4. GCF and LCM
    - Applying Formulas and Logics
    - Counting Possibilities
5. Divisibility
    - Divisibility Rule Implementation
    - Divisibility + Sequences
    - Divisibility + (Properties of Numbers + Modular Arithmetic)
    - Divisibility + Combinations
6. Alphabet Problems

<u>Even and Odds:</u>

<u>Properties:</u>

1. Addition
    - e±e=e, e±o=o, o±o=e
2. Multiplication
    - e*e=e, e*o=e, o*o=o
3. Common representations:
    - even: 2n
    - odd: 2n-1 || 2n+1
4. The sum of two consecutive numbers is always odd (Addition rule)

<u>Typical Problems:</u>

Problem 1: How many different pairs (m,n) can be formed using numbers from the list of integers {1,2,3,..., 20} such that $m<n$ and m+n is even?

Prerequisites: m+n is even such that m and n must be of the same parity.

<u>Solution 1 (most efficient): Using Combinatorics</u>

If m and n are odd and of the same parity then we can choose two numbers out of all of the numbers from one parity and assign the number of greater value to n and the other to m. We form two cases, choosing two numbers out of evens and choosing two numbers out of odds:

Evens: There are 10 even numbers total: 2C10=10*9/2=45;
Odds: There are 10 odd numbers total: 2C10=10*9/2=45;
Total = 90;

<u>Solution 2 (second most efficient): Using the symmetry principle</u>

Even: $C(n<m)=C(n>m)$; Thus, $C(m<n)=(10*10-C(n=m))/2=100-10/2=45$;

Odd: $C(n<m)=C(n>m)$; Thus, $C(m<n)=(10*10-C(n=m))/2=100-10/2=45$;

Solution 3: (Least efficient but most generalized): Use organized counting

N: M:
1, 3->19 9
2, 4->20 9
3, 5->19 8
4, 6->20 8
…
17, 19 1
18, 20 1
= 45*2=90

Problem 2: For any two positive integers a and b, find the number of pairs (a,b) such that a-b is odd and 24<=a<b<=40?

Solution: Use organized counting to solve

Given that a-b is odd, a and b are of different parities; thus,
n, m:
1, 2->20; 10
2, 3->19; 9
3, 4->20; 9
4, 5->19; 8
5, 6->20; 8
...
18,19; 1
19,20; 1
Total 45*2+10=100;

Problem 3: Jensen is trying to get a result of 28 by placing operation + or - between numbers in the expression 1_2_3_4_5_6_7_8_9. Is it possible for him to make it? Why?

Solution: Addition and subtraction will always produce the same parity as long as the terms remain the same. If we add all the numbers together, the result is 45, thus, no matter how we place the signs, the result will always be odd and cannot be 28.

Problem 4: For any two positive integers m and n, find the number of pairs (m,n) such that mn+m is odd and 1<=m<n<=20?

We can convert mn+m to m(n+1), thus m and (n+1) must both be odd.

Solution (most efficient): Use combinations: For, we are essentially choosing two numbers from a set of odd numbers and assigning the smaller one to m.

Thus, there are 2C10=45 possibilities;

Problem 5: For any two positive integers m and n, find the number of pairs (m,n) such that mn+n-m is even and 1<=m<=n<=20?

We can convert mn+n-m into n(m+1)-m. Given that the result is even, we know that n(m+1) and m are of the same parity. If they are both odd, then m will be odd and m+1 will be even resulting in n(m+1) being even. This case is thus invalid. If both terms are even, then m will be even, and n will thus need to be even as well.

We can use organized counting to solve from here:

2,2->20 10

4,4->20 9

...

18,18->20 2

20, 20 1

Total = (10+1)10/2=55;

Primes:

Properties and useful facts:

1. 25 primes from 1-100
2. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
3. If the product of any number of prime numbers is even then one of the prime numbers must be two.
4. If the sum of two prime numbers is odd then one of the numbers must be two.

Types of Problems:

1. Largest Prime Divisor
2. Implementing Properties + Even-Odd rules
3. Prime factorization

Largest Prime Divisor Problems:

Problem 1: What is the greatest prime divisor of 13!-12!+11!?

11!(13*12-12+1)=11!(143); The largest prime factor of 11! is 11 and the largest prime factor of 143 is 13; Thus, the greatest prime divisor of the expression is 13.

Implementing Properties + Even-Odd Rules

Problem 2: If the sum of three prime numbers is 31 and their product is 1001, what is the largest of the three prime numbers?

If the product of three prime numbers is 1001 then we can prime factorize 1001 to obtain a result. 1001=11*13*7. The largest of the prime numbers is 13.

Problem 3: Find all 4-tuples (a,b,c,d) of primes satisfying both of the equations abcd+ |c-d| = 2020 and abcd+ |a-b| = 2116

There are two cases, both terms are even or both terms are odd. The case in which both terms are odd is invalid given that, in order for both terms to be odd, c and d must be of different parity. If c and d are of different parity, then abcd will be even resulting in an invalid case. The only valid case is both even. If both terms are even, then either a,b are both of the same parity, and c,d are both the opposite parity of c and d.

Case 1: (a,b) = 2,2:

This case is invalid given that we can derive the following statement: |c-d|=-96;

Case 2: (c,d) = 2,2;

|a-b|=96 We can substitute the equation into the statement abcd + |a-b| = 2116:
abcd=2020; ab=505; Now we can prime factorize => a*b=5*101; thus, (a,b)=(5,101),(101,5);

tuples 1: (5,101,2,2),(101,5,2,2);

<u>Prime factorization</u>

Problem 4: Express the prime factorization of 17!

Solution (Formal Solution): We can first represent 17! as all of the prime numbers less than 17 with unknown bases:

2,3,5,7,11,13,17

Now, to determine the exponents of each base we can determine the number of powers of 2 and sum them up to determine the exponent of 2.

17/2=8; 17/4=4; 17/8=2; 17/16=1; Thus, the exponent of 17 is 15. We repeat this process for each prime number to determine the final factorization, keeping in mind that the larger the base becomes, the less work that will be required.

Solution (Most Efficient Solution): Use repeated division to simulate the process explained in the previous solution.

First, we represent the bases the same way as the previous solution:

2,3,5,7,11,13,17

To determine the exponents of the bases, we can use repeated division as follows:

2|17
2|8
2|4
2|2
2|1

We then sum up the results 8+4+2+1=15 to get the exponent. Repeating the process, we can determine that the prime factorization is thus, 2^15*3^7*5^3*7^2*11*13*17.

Factors + Multiples:

Properties + Formulas:

1. Number of factors of N = (P1+1)(P2+1)(P3+1)(P4+1)...(Pn+1) where P1->PN are all the exponents found in the prime representation of N.
2. Number of multiples of A <= N = floor(N/a)
3. If all the exponents of a number N are divisible by (a) then (a) root(N) is a whole number.
4. Sum of factors: $S = \frac{a^{p+1} - 1}{a-1} \times \frac{b^{q+1} - 1}{b-1} \times \frac{c^{r+1} - 1}{c-1}$

Types of problems:

1. Basic Implementation
2. Set theory divisibility
3. Algebra and Divisibility

Basic Implementation

Problem 1: How many perfect squares are factors of 400?

400=2^4*5^2. For a number to be a perfect square it has to have even factors such that the number can contain 2^0,2^2, or 2^4. Also, the number could contain 5^0 or 5^2; Meaning that there are a total of 3*2=6 numbers.

Problem 2: How many even factors does 210 have?

210 = 2*3*5*7. The factor must contain two and the rest have two options each. Meaning that there are a total of 2^3=8 options.

Problem 3: What is the prime factorization of the smallest positive integer that has exactly 31 factors?

There is essentially a multiplication statement that has a product of 31, so the only possibility is one prime factor to the power of 30. This means that the smallest possible number is 2^30.

Problem 4: What is the smallest positive integer that has 18 factors?

There are four cases given the following statement: (18), (2,9), (3,6), (2,3,3)

The solution will then be the number with the lowest sum of exponents, which we can identify to be (2,3,3); Such that the smallest possibility is 2^2*3^2*5^1=36*5=180.

Problem 5: How many two-digit numbers have exactly three factors?

If a number has exactly three factors, it means that the number must be a perfect square. Additionally, the prime factorization must only consist of a single prime number whose power makes it a perfect square. Their valid numbers are the following, 5^2, 7^2. Such that there are 2 possibilities.

Problem 6:

B2 For each positive integer n, define $\varphi(n)$ to be the number of positive divisors of n. For example, $\varphi(10) = 4$, since 10 has 4 positive divisors, namely $\{1, 2, 5, 10\}$.

Suppose n is a positive integer such that $\varphi(2n) = 6$. Determine the minimum possible value of $\varphi(6n)$.

We can determine the possible scenarios for 2n and then multiply by 3 to determine 6n.

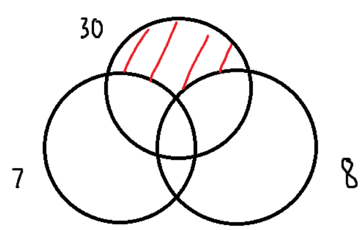
Where P is any odd prime number !=3
2n = 2*P^2 => 6n has 12 factors
2n=2*3^2 => 6n has 8 factors
2n =2^5 => 6n has 12 factors
2n = 2^2*P => 6n has 12 factors
2n=2^2*3 => 6n has 9 factors

Thus, the minimal number of factors for 6n is 8.

Set Theory Divisibility

Problem 6: How many positive integers less than 2011 are divisible by 5 and 6 but not divisible by 7 or 8?

We can construct a diagram using set theory:



The target area is the number of numbers divisible by thirty but not 7 or 8:

floor(2010/30)-floor(2010/210)-floor(2010/120)+floor(2010/840)=67-9-16+2=44

LCM/GCF Problems:

Formulas:

1. GCF: To find the greatest common factor (GCF) between numbers, take each number and write its prime factorization. Then, identify the factors common to each number and multiply those common factors together.

2. LCM: Prime factorize the numbers and determine the largest number of times each prime factor occurs between both numbers. Now multiply these values to get the LCM
3. GCF(a,b)*LCM(a,b)=a*b
4. GCF(a^2,b^2)=GCF(a,b)^2
5. GCF(a,b)=1, and (a*b) is a perfect square, then a and b are both perfect squares

Problem 1: Janice and Kiera begin jogging around a track, starting at the finish line and going the same direction. Janice completes a lap every 78 seconds, while Kiera takes 90 seconds. At the end of their workout, they cross the finish line together in a whole number of minutes. What is the minimum number of minutes that they could have run for?

LCM(90,78)=1170. To make the time a whole number of minutes, the number of seconds must be divisible by 60. We are missing a 2 so we must multiply 1170 by 2 resulting in 2340=39 mins.

Problem 2: Ken gets his haircut every 20 days. Larry gets his cut every 26 days. Ken and Larry get their hair cut on the same Tuesday. What day of the week is it the next time they get their hair cut on the same day?

The next time they get their hair cut, will be LCM(20,26)=260 days into the future. Dividing this by 7 we get a remainder of 1 significant that the next day on which they will both get their hairs done will be Wednesday.

Problem 3: How many different pairs of whole numbers have a greatest common factor of 4 and a lowest common multiple of 4620?

4*(11^(0,1)*7^(0,1)*3^(0,1)*5^(0,1)),4*(11^(1,0)*7^(1,0)*3^(1,0)*5^(1,0))=16 with counting arrangements. It is safe to assume that the question isn't asking for the result with arrangements, such that there are 16/2=8.

<u>Fraction Number Theory:</u>

Problem 1:

4. (a) Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 100$, that is closest to, but not equal to, $\frac{3}{7}$.

|(p/q)-(3/7)|=|(7p-3q)/(7q)|

Minimize the value of the expression above, we must maximize the denominator and minimize the numerator;
We know $q<99$, so we can test values of q and stop when 7p-3q=1;

If $q = 99$, $7p - 3q = 7p - 297$, which cannot equal ± 1 since the nearest multiple of 7 to 297 is 294.
If $q = 98$, $7p - 3q = 7p - 294$, which cannot equal ± 1 since $7p - 294$ is always divisible by 7.
If $q = 97$, $7p - 3q = 7p - 291$, which cannot equal ± 1 since the nearest multiple of 7 to 291 is 294.
If $q = 96$, $7p - 3q = 7p - 288$, which equals -1 if $p = 41$.

Thus, the only other possibilities are when q>96 and 7p-3q>1. Thus, because 2/(99*7)>1/(96*7)
Thus, 41/96 is the closest number to 3/7;

Key ideas: Use absolute values to express the problem statement and use fractional numerator and denominator logic to solve.

Divisibility Problems:

Divisibility Rules + Common Properties:

(3,9): If a number is divisible by one of the following numbers, then the sum of its digits must also be divisible by that number.
(2,4,8,16…) If a number is divisible by one of the following numbers, then the number formed by the n rightmost digits must also be divisible by that number. (where n is the power of 2: 2^n)
(5): If a number is divisible by 5, then the number must end with 0 or 5.
(11): If a number is divisible by 11,then its alternating sum must be divisible by 11.
(7,13,17,19): If a number is divisible by the following, then the result obtained by multiplying (-2,+4,-5,+2 respectively) by its unit digit and adding it to the remaining number. (483=48-3*2=42. Thus, 483 is divisible by 7)

Lemmas

1. If a,b, and c are integers such that a|bc and GCD(a,b)=1, then a|c
2. If a,b, and c are integers such that a|c, b|c and GCD(a,b)=1, then ab|c
3. If a,b are in range (1,n) and a^2|b^2, then a|b

Typical Questions:
1. Divisibility Rule Implementation
2. Divisibility + Sequences
3. Divisibility + (Properties of Numbers + Modular Arithmetic)
4. Divisibility + Combinations

Divisibility Rule Implementation:

Problem 1: What is the smallest 4-digit number that is divisible by 2,3,4,5,6,8,9, and 10?

Problem 2: What is the smallest 5-digit integer divisible by both 8 and 9?

Problem 3: The digits 1,2,3,4, and 5 are each used once to construct a five-digit number PQRST. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

Problem 4: If a and b are positive integers, neither of which is divisible by 10, and if a*b = 10e5 then what is the sum a+b?

Problem 5: How many palindromes greater than 10000 and less than 100000 are multiples of 18?

<u>Divisibility + Sequences</u>

Problem 1 (Fibonacci Sequence): The list 11,20,31,51,82 is an example of an increasing list of five positive integers in which the first and the second integers add to the third, the second and the third add to the fourth, and the third and fourth add to the fifth. How many such lists of five positive integers have 124 as the fifth integer?

Problem 2 (Repeating Group): The smallest positive integer n for which n(n+1)(n+2) is a multiple of 5 is n=3. All positive integers, n, for which n(n+1)(n+2) is a multiple of 5 are listed in increasing order. What is the 2018th integer in the list?

<u>Divisibility + Properties of a Number</u>

Problem 1 (Modular Arithmetic): The ages of 3 sisters are M,M+2,M+4 where M is a positive integer. Both the numbers 5 and 7 divide the sum of their ages. What is the smallest possible value of M?

<u>Divisibility + Combinations</u>

Problem 1: The number 2018 is used to create six-digit positive integers. These six-digit integers must contain the digits 2018 together and in this order. For example, 720186 is allowed, but 2091313 is not. How many of these six-digit integers are divisible by 9?

Problem 2: Lynne forms a 7-digit integer by arranging the digits 1,2,3,4,5,6,7 in random order. What is the probability that the integer is divisible by 11?

Problem 3:

25. Brady is stacking 600 plates in a single stack. Each plate is coloured black, gold or red. Any black plates are always stacked below any gold plates, which are always stacked below any red plates. The total number of black plates is always a multiple of two, the total number of gold plates is always a multiple of three, and the total number of red plates is always a multiple of six. For example, the plates could be stacked with:

- 180 black plates below 300 gold plates below 120 red plates, or
- 450 black plates below 150 red plates, or
- 600 gold plates.

In how many different ways could Brady stack the plates?

(A) 5139 **(B)** 5142 **(C)** 5145 **(D)** 5148 **(E)** 5151

If we represent the plates using variables a,b,c respectively, we can form the following equations;

We can represent the conditions given through the following equation:

2a+3b+6c=600;

By manipulating the equation to we can determine the properties of the variables:
$2a=600-3b-6c$; => $2a=3(200-b-2c)$ => $2a/3=200-b-2c$;

We know that $200-b-2c$ is an integer such that a must be divisible by 3 (given that 2 isn't divisible).

We can similarly manipulate the equation in terms of b and determine that b is divisible by 2. Given this, we can divide the entire equation by 6, maintaining the fact that each term is an integer:

$2a/3+3b/2+c=100$;

We are essentially dividing 100 identical 1's among three groups, where a group can have zero 1's. We can use the distinguishability formula => $(n+m-1)C(m-1) = 100+(3-1)C(3-1) = 102*101/2=51*101$.

Only one of the given options has a one in its unit digit, such that the answer must be 5151.

Problem 4: How many ordered pairs (a,b) of integers are there such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{8}$? Note that if $x \neq y$ then the ordered pair (x,y) is different from the ordered pair (y,x). (Hint: note that integers can be either positive, or negative).

Strategy: Reduce the equation into a form in which the product of two terms is equal to a constant; After, count the factors of the constant, taking into account that both a and b cannot be equal to 0;

$8a+8b=ab$; => $ab-8a-8b=0$; => $a(b-8)-8(b-8)=64$; => $(a-8)(b-8)=64$; $64=2^6$; Thus, there are 6+1 positive factors; Thus, there are 7 positive ordered pairs;

For the negative pairs, the only expectation is $-8*-8$; because the result will yield $a=0$, and $b=0$ which are invalid values for both a and b. There are 6 negative pairs.

In total, $7+6=13$ pairs of integers for a and b;

Problem 5: In 1991 the population of a town was a perfect square. Ten years later, after an increase of 150 people, the population was 9 more than a perfect square. Now, in 2011, with an increase of another 150 people, the population is once again a perfect square. Which of the following is closest to the percent growth of the town's population during this twenty-year period?

Solution:

We can represent the problem statement using a basic diagram:

a^2 => p
$b^2+ 9$ => p+150
c^2 => p+300

We can create the statement: $b^2+9=a^2+150$ => $(b-a)(a+b)=141$.
$(b-a),(a+b)=(1,141),(3,47)$
For case 1, $b=71$, $a=70$. Since 70^2+300 isn't a perfect square, this case is invalid.

For case 2, b=25, a=22. 22^2+300=784=28^2.
The percent increase is 300/484 or ~0.619.

Problem 6: An Anderson number is a positive integer k less than 10 000 with the property that k 2 ends with the digit or digits of k. For example, 25 is an Anderson number because 625 ends with 25, but 75 is not an Anderson number because 5625 does not end with 75. If S is the sum of all even Anderson numbers, what is the sum of the digits of S?

Strategy: Start with single digit num bers and recursively work upwards from their.

Out of the first 9 numbers: 1,2,3, …. 8,9 -> Only 6 produces a number with a units digit of 6 when squared.
Next, the 2 digit numbers can be 16,26,36… 96 (they must always end in 6 since any other number would result in the units digit being invalid) After testing all the 9 numbers, 76 is the only one that works.

Next, the 3 digit numbers can be 176,276, … 976. Out of these, 376 is the only one that works.

Finally, the 4 digit numbers can be 1376, … 9376. Out of these, 9376 is the only one that works.

Thus, the sum is 9834.

Divisibility + Algebra / Functions:

Problem 1:

C3. Let n be a positive integer. A row of $n+1$ squares is written from left to right, numbered $0, 1, 2, \ldots, n$, as shown.

0	1	2	$\cdots$	n

Two frogs, named Alphonse and Beryl, begin a race starting at square 0. For each second that passes, Alphonse and Beryl make a jump to the right according to the following rules: if there are at least eight squares to the right of Alphonse, then Alphonse jumps eight squares to the right. Otherwise, Alphonse jumps one square to the right. If there are at least seven squares to the right of Beryl, then Beryl jumps seven squares to the right. Otherwise, Beryl jumps one square to the right. Let $A(n)$ and $B(n)$ respectively denote the number of seconds for Alphonse and Beryl to reach square n. For example, $A(40) = 5$ and $B(40) = 10$.

(a) Determine an integer $n > 200$ for which $B(n) < A(n)$.

(b) Determine the largest integer n for which $B(n) \leq A(n)$.

(a)

B(n) is minimized when n is congruent to 0 mod 7. Thus, n is in form 7k

B(n) is maximized when n is congruent to 7 mod 8. Thus, n is in form 8j+7;

Thus, n = 231 is valid.

(b)

If B(n) is minimized to ensure the inequality, then n = 7k.

If A(n) is maximized to ensure the inequality, then n= 8j+7.

If 7k=8j+7; 7|j

Thus, when n=56i+7; B(n)=7i+7; A(n)=8i+1;

For B(n)<=A(n); 7i+7<=8i+1; i<=6; Thus, the maximum value of n for which B(n)<=A(n) is **343**;

Set Problems:

Problem 1: How many different integers can be expressed as the sum of three distinct members of the set 1,4,7,10,13,16,19?

Subtracting 10 from each number in the set, and dividing the results by 3, we obtain the set $\{-3,-2,-1,0,1,2,3\}$. It is easy to see that we can get any integer between -6 and 6 inclusive as the sum of three elements from this set, for a total of 13 integers.

Factorial Number Theory

Problem 1:

8. Determine all triples (a, b, c) of *positive* integers such that $a! = 4(b!) + 10(c!)$.

Note: If n is a positive integer, the symbol $n!$ (read as "n factorial") is used to represent the product of the positive integers from 1 to n; that is,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

For example, $5! = 5(4)(3)(2)(1)$.

Here, we use case work to simplify the problem.

We know a>b,c so we divide it into cases about the relationship between b and c:

If b=c:

(a,b,c)=(14,13,13)

If b>c:

Now we divide by b! which is quite convenient here;

a!/b! = 4+10/(b(b-1)*...(c+1))

Either b=c+1 (the denominator is a single number) => b=2,5,10; => (a,b,c)=(6,5,4)

Or b=c+2 (the denominator is the product of two consecutive numbers) => this doesn't work here

If b<c: we do the opposite and find that (a,b,c)=(4,2,1)

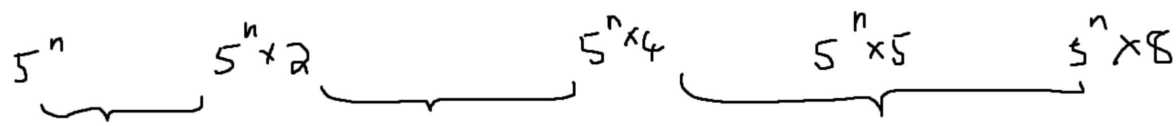
Exponents:

The number 5^{867} is between 2^{2013} and 2^{2014}. How many pairs of integers (m, n) are there such that $1 \leq m \leq 2012$ and

$$5^n < 2^m < 2^{m+2} < 5^{n+1}?$$

(A) 278 **(B)** 279 **(C)** 280 **(D)** 281 **(E)** 282

This problem involves the gaps between powers of 5 and 2.



Each of the drawn gaps indicates a position in which a power of two must lie. You can see, from this, that each pair (m,n) will have either 2 or 3 powers of 2 between the consecutive powers of 5.

There are 2*867 guaranteed gaps. There are 2013 gaps in total. Thus, the answer is 2013-2*867

Base Manipulation:

Important properties:

1. Base 2 can be directly converted to base 8 by splitting 1010111 into sections of 3. This is because 8=2^3; From this we get three different digits; 111=7,010=2,1=1 which translates to 127 in base 8; We can use this method with any two numbers that are in the form a^n,a^n*m where a,n,m are whole numbers. (These prime factors must be equal.) Valid pairs include (2,4); (3,9); etc.

2. If you divide a base n number by n then you are effectively shifting the number 1 place to the left (There may be remainders if the removed value is not zero).

3. This property can be expanded. For example, if you are trying to find how many numbers in a certain range have a binary representing ending with zero, then you are effectively searching for the multiples of 2 within that range.

Types of problems:

1. Basic Conversion
2. Finding base value
3. Determining range using properties

Basic Conversion:

Problem 1: Evaluate √61 base 8) as a number in the decimal system.

Target: answer;
Strategy converts value inside the square root to decimal and solve.

61 base 8= $8^0*1+8^1*6=49$
√49)=7;

Problem 2: Find the base 2 number that is equivalent to 57 base 8.

Target: base 2 equivalent;
Strategy: Use property discussed to determine the solution;

We can simply convert each of the digits to binary representation as if they were in decimal representation;

7 = 111 and 5 = 101;
Number = 101111;

<u>Finding Base Value:</u>

Problem 3: N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Target: Smallest base b;
Strategy: Use the basic conversion formula to set up a quadratic equation and solve;

777 base b = $b^0*7+b^1*7+b^2*7=7b^2+7b+7=a^4=7(b^2+b+1)=a^4$;
for b to be minimal, then b^2+b+1 needs to be equal to 7^3;
$b^2+b+1=343 \Rightarrow b^2+b-342 \Rightarrow (b-18)(b+19)=0$;
Therefore, the minimal value for b is 18;

Problem 4: For some positive integer k, the repeating base-k representation of The (base-ten) fraction 7/51 is 0.23.k=0.232323232323...k. What is k?

Target: base k;
Strategy: Use conversion formula to set up quadratic equation and solve.

x=0.23...;
$x*k^2=23.23...$;
$(k^2-1)*x=23$;
7/51x=23 base $k/(k^2+1) = > 7k^2-7=51(2k+3)$;
$7k^2-102k-160=0$;
$(7k+10)(k-16)=0$;
k=16;

<u>Determining range using properties:</u>

Problem 5: How many 4-digit positive integers have zeros in both of the two last places of their binary expansion?

Target: Range of binary numbers ending with 00;
Strategy: Use information given about the properties of the number to determine the range;

For a number to end in 00 it means that it is able to be shifted to the right twice without any remainder. A shift signifies multiplying by 2 in binary such that all numbers ending in 00 must be divisible by 4;

4 digit numbers divisible by 4= (9996-1000)/4+1=2249+1=2250

Problem 6: Let n be a positive integer. Define a sequence
a1, a2, a3, . . . where a1 = n and floor((am-1)/3); for all integers m = 2. The sequence stops when it reaches zero. The number n is
said to be lucky if 0 is the only number in the sequence that is divisible by 3. For example, 7 is lucky, since a1 = 7, a2 = 2, a3 = 0, and none of 7, 2 are divisible by
3. But 10 is not lucky, since a1 = 10, a2 = 3, a3 = 1, a4 = 0, and a2 = 3 is divisible by 3. Determine the number of lucky positive integers less than or equal to
1000.

Target: Number of 'lucky numbers' under 1000;
Strategy: Use the properties of bases to help solve for the max value and determine a pattern;

1000 base 10 = 1101001 base 3;
For a base 3 number not to be divisible by 3, it must not include a zero.
Therefore, 222222 is the largest base 3 value that satisfies the restrictions.

From this number, we can deduce that the total number of 6 digit base 3 the number possible is 2^6 because there are two options for each digit. We can repeat this process for all the number of digits until we are left with the expression => 2^6+2^5+2^4+2^3+2^2+2^1=126;

Algebraic Number Theory:

Strategy:

1. Trying to simplify an expression such that the a/c structure is formed where a is a constant and c is a single variable.
2. Reducing the degree of the numerator with polynomial division
    a. **Lets say we have: (n^2)/n-1 is an integer**. Then, after performing polynomial division, you get n+1+(1/(n-1)) is an integer, which means that n-1 has to divide 1. So, n-1=1,-1 and n = 0 , 2.
    b. **Lets say we have: (n^2+1)/(2n-1) is a natural number**. We can multiply the numerator by 2 while maintaining the fact that the expression reduces to a natural number. (2n^2+2)/(2n-1) is a natural number. So, (2n+4)/(2n-1) is a natural number and 5/(2n-1) is a natural number. Then, 2n-1 has to divide 5. So, 2n-1 is either -5, -1, 1, 5, so n is either -2, 0, 1, 3, which all work.

Problem 1: COMC 4B 2022

Problem 2: OLYMPIAD.CA, NUMBER THEORY 4, 2022

Sequences:

Types of Problems:

1. Arithmetic Sequence
2. Geometric Sequence
3. Infinite Sequence
4. Fibonacci Sequences
5. Miscellaneous Sequences

Arithmetic Sequence Formulas:

an=a1+(n-1)d; n=(an-a1)/d+1;
Sum of sequence=((a1+an)n)/2;
Middle Term = Sum/n, aq+ap=am+an ⇔ p+q=m+n
Example: a1,a2,a3,a4,a5; => a1+a5=a2+a4=a3*2

Geometric Sequence Formulas:

an=last term; an=a1*r(n-1);
Sum of sequence: a1*(1-r^n)/(1-r);
Sum of an infinite geometric sequence: a1/(1-q)

Arithmetic Sequence Properties:

1. If n is odd, then the middle term of the elements is equal to the average;
2. an is the average of an-1 & an+1;
3. The sum of consecutive integers has an odd factor
5. An arithmetic sequence consisting of consecutive odd integers has a sum of n^2;

Arithmetic Sequence Problems:

Problem 1: In the five-sided star shown, the letters A, B, C, D, and E are
replaced by the numbers 3, 5, 6, 7 and 9 although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?

Target: The middle term of the arithmetic sequence;
Strategy: Use the 1st property to solve for the middle term;
Every number is added twice such that the total sum= 2(3+5+6+7+9)=60;
The mean=60/5=12;

Problem 2: How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

Target: Degree measures are distinct positive integers in arithmetic progression?
Strategy: Determine the middle term through the first property and count the number of least possible values of the angles;

Middle Term=180/3=60;
The smallest angle ranges from 1 to 59;
Therefore there are 59 possible angles.

Problem 3: A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

Target: The total sum of the arithmetic progression subtracted by the intersection.
Strategy: Find the arithmetic sum and subtract intersection;
Arithmetic sum=((20+3)18)/2=9*23=207;
Intersection = 17*2 because there are 17 points of intersection of length 1+1=2;
Answer=207-34=173;

Problem 4: How many sets of two or more consecutive positive integers have a sum of 15?

Strategy: Casework made efficient by using factors

The consecutive positive integers have a sum of 15, then their average (which could be a fraction) must be a divisor of 15. If the number of integers in the list is odd, then the average must be either 1, 3, or 5, and 1 is clearly not possible.

If the number of integers in the list is even, then the average will have a frac 1/2. The only possibility is 15/2, from which we get:

15 = 7 + 8
Thus, there are 3 possible ways.

Problem 5: The non-negative integers a, b, c, d and e form an arithmetic sequence. If their sum is 440, what is the largest possible value for e?

To maximize the largest value in an arithmetic sequence the terms must have the maximum possible difference. For this to hold true, a, the least value, must be the lowest possible value, 0; We know that c, the middle term, is equal to the median. Thus, c=440/5=88; We also know that a+e=b+d=2c; Thus, 0+e=176; e=176;

Problem 6: A sequence starts out with one 5, followed by two 6s, then three 7s, four 8s, five 9s, six 5s, seven 6s, eight 7s, nine 8s, ten 9s, eleven 5s, twelve 6s, and so on. (You should notice that only the five digits from to are used.)

Determine the 2022 digit in the sequence.

Target the 2022 digit in the sequence:
Strategy: Use the sum formula to determine the section that the 2022nd digit resides;

((1+n)n)/2=2022;
$n+n^2=4044$;
n(n+1)=4044;
sqr(4044)=63.59...
Therefore n=63 and 2022 resides in the 64's row;
The pattern 5,6,7,8,9 is of length 5. So the 64's row represents 64 mod 5 = 4
so the number in the 64th section is the fourth that appears.

The 2022 digit is 8;

Problem 7: Sam and Delilah are reading different books. Today, Sam and Delilah read one chapter in their respective books, and they each read more than one page. Interestingly, they read the same number of pages, but the sum of the page numbers for the chapter Sam read was 880, and the sum of the page numbers for the chapter Delilah read was 1008. How many pages did Sam read today?

(a1+an)n=1760;
(b1+bn)n=2016;
n is a common factor so we can prime factorize both 2016 and 1760 to find their common factors.
$1760=2^5 * 5 * 11$
$2016=2^5 * 9 * 7$
The sum of two consecutive terms must be odd, and a1+an is equivalent to the sum of the middle two terms. Thus, the only possible value for n is 32.

Problem 8:

8. The number 18 *is not* the sum of any 2 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 2 different ways, since $5+6+7=18$ and $3+4+5+6=18$. Determine a positive integer less than 400 that *is not* the sum of any 11 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 11 different ways.

If a number N is equal to the sum of an odd number (2k+1) of consecutive numbers:
N = (2k+1)a (where a is the middle number)
If a number N is equal to the sum of an even number (2k) of consecutive numbers:
N= (2k)b-k = k(2b-1); (where b is the middle [right] number) Thus, N is divisible by k but not 2k;

Making a chart of all the values below 10:

m	N at least	Property of N
2	3	Divisible by 1, not by 2
3	6	Divisible by 3
4	10	Divisible by 2, not by 4
5	15	Divisible by 5
6	21	Divisible by 3, not by 6 (ie. divisible by 3, not by 2)
7	28	Divisible by 7
8	36	Divisible by 4, not by 8
9	45	Divisible by 9
10	55	Divisible by 5, not by 10 (ie. divisible by 5, not by 2)

In order to maximize the value of N, we should maximize the number of values m we use:
If n is divisible by 5*7*9 = 315, N will have the largest number of consecutive divisors: 2,3,5,6,7,9,10;

Because 315 produces the most number of consecutive divisors and N is below 400, N=315.

m=14,15, 21, 18;
Thus, N = 315 is valid.

Problem 9:

4. In a *sumac sequence*, $t_1, t_2, t_3, \ldots, t_m$, each term is an integer greater than or equal to 0. Also, each term, starting with the third, is the difference of the preceding two terms (that is, $t_{n+2} = t_n - t_{n+1}$ for $n \geq 1$). The sequence terminates at t_m if $t_{m-1} - t_m < 0$. For example, 120, 71, 49, 22, 27 is a sumac sequence of length 5.

   (a) Find the positive integer B so that the sumac sequence $150, B, \ldots$ has the maximum possible number of terms.

   (b) Let m be a positive integer with $m \geq 5$. Determine the number of sumac sequences of length m with $t_m \leq 2000$ and with no term divisible by 5.

(b)

We can work backwards to make the numbers easier to work with. We let t[m]=x and t[m-1]=y. The sequence must be at least 5 numbers long. Thus, the first five terms can be represented as:

2x+3y, x+2y, x+y, y, x;

Next, we test the valid parities of x,y such that no term is divisible by 5

x	y	$x+y$	$x+2y$	$2x+3y$
1	1	2	3	0
1	2	3	0	
1	3	4	2	1
1	4	0		
2	1	3	4	2
2	2	4	1	0
2	3	0		
2	4	1	0	
3	1	4	0	
3	2	0		
3	3	1	4	0
3	4	2	1	3
4	1	0		
4	2	1	3	4
4	3	2	0	
4	4	3	2	0

Case 1: (x,y ≡ 1,3 [mod 5]):

We know that x<=2000 and that y<x;
If x=1996 and y=1993, 1988, … 3 (399 possibilities)
If x=1991 and y=1988, … 3 (398 possibilities)
…
If x=6 and y=3 (1 possibilities)
Total Possibilites = 399+398+...+1

Case 2: (x,y ≡ 2,1 [mod 5]):

If x=1997 and y=1996, 1991, … 1 (400 possibilties)
…
Total Possibilities = 400+399+398+...+1

Case 3: (x,y ≡ 3,4 [mod 5]):

If x=1998 and y=1994, 1989, …+4 (399 possibilities)
…
Total Possibilities = 399+398+...+1

Case 4: (x,y ≡ 4,2 [mod 5]):

If x=1999 and y=1997,1995,...,2 (400 possibilities)
…
Total Possibilities = 400+399+398+...+1

Thus, the overall number of valid possibilities = **320,000**

Problem 10:

Problem 24

A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 31 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 33 \qquad \textbf{(E)}\ 34$

Solution

Geometric Properties:

1. The middle term is equal to the mean of the sequence assuming n is odd.
2. an is the average of an-1 & an+1;

Geometric Problems:

Problem 1: Find a20 of a geometric sequence if the first few terms of the sequence are given by -1/2 , 1/4 , -1/8 , 1 / 16?

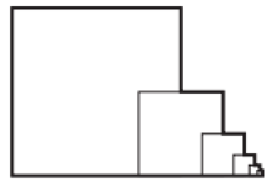
Target: a20
Strategy: Substitute known values into formula and solve;
Formula: an=last term; an=a1*r^(n-1);
r=(1/4)/(-1/2)=-1/2;
a20=-1/2*-1/2^19=(1/2)^{2}0=1/2^{2}0;

Problem 2: Find the sum of a geometric sequence with a first term 1 and a last term 243 with a ratio of 3 between each element and 6 elements total.

Target: Sum;
Strategy: Substitute values into formulas;
Formula: Sum of sequence: a1*(1-r^n)/(1-r);
1*(1-3^6)/1-3)=-728/-2=364;

Problem 3 (infinite geometric series):

Starting with a square of side length 1, a square of side length $\frac{1}{2}$ is drawn so that it is bisected by a side of the original square, as shown. This process is repeated with a square of side length $\frac{1}{4}$ bisected by the square of side length $\frac{1}{2}$, and so on without end. What would be the area of such a figure, when generated as described? Express your answer as a common fraction.



Strategy: Identify an infinite geometric series and apply the formula

The areas of the rectangles formed follow are as follows:
1/8,1/32,1/128,1/512,...
The starting term is 1/8 and the quotient is 1/4. Inserting these values into the formula: (⅛)/(1-¼)=⅛*4/3=1/6;
Adding the initial area of 1: 1+⅙=7/6;

Combinatorics

<u>Probability:</u>

<u>Primary Types of Problems:</u>
1. Coin Flips
2. Dice Rolls
3. Taking Items out of a group
4. Expected Value

<u>Strategies to Solve:</u>
1. Using Pure Probability
2. Using Combinations
3. Using Tree Diagrams
4. A combination of the strategies mentioned prior

Summary: In each question, I will attempt to summarize the solution using as many methods as possible, particularly on more advanced problems. Note that some methods are not possible, depending on the problem type. I will summarize this unit based on different question types and difficulty levels;

<u>Coin Flips (Independent Probability):</u>

Coin Flips: (Extremely Easy + Independent): A coin is tossed 5 times. What is the probability that no 2 heads show up consecutively?

Strategy (combining probability with counting/combinations):

The only possible positions are:
HTHTH
THTHT

Thus the final probability is (½)^5*2 = <u>1/16</u>;

Coin Flips (Relatively Easy + Independent): If you flip 5 coins what is the probability of getting at least 2 heads?

Strategy 1 (most efficient): Using complementary probability and combining probability and combinations; (Note that when the phrasing <u>at least</u> is used, complementary probability is often useful)

Instead of considering the probability of getting at least 2 heads, we can consider the probability of getting less than two heads and get its complement:

Case 1 (1 head): (½)^5*1C5=5/32;
Case 2 (0 heads): (½)^5*0C5=1/32;
Sum = 3/16; Complement = 1-3/16=13/16;
Solution = <u>13/16</u>;

Strategy 2 (Brute force calculation using combinations):

Case 1 (2 heads): 2C5/(1C2)^5
Case 2 (3 heads): 3C5/(1C2)^5
Case 3 (4 heads): 4C5/(1C2)^5
Case 5 (5 heads) 5C5/(1C2)^5
Sum = <u>13/16</u>;

<u>Dice rolls (Independent Probability)</u>

Dice rolls (extremely easy + independent): (1) You throw two dice. What is the probability that the sum of the two dice is 8? (2) You throw two dice. If there are exactly 6 ways to produce the sum. What is the sum?

(1) Strategy: Use counting + probability:

Possible pairs: (2,6), (3,5), (4,4); Taking into account order: 2*2+1=5;

(2) Strategy: Use counting + probability+logic:

We can go through possible sums until we find a sum with 6 ways to produce it:

Sum = 5: (1,4),(2,3) Possabilities =4;
Sum = 6: (1,5),(2,4),(3,3) Possibilities = 5;
<u>Sum = 7: (1,6),(2,5),(3,4) Possabilities = 6;</u>

Dice rolls (easy + Independent): What is the probability of rolling the same number exactly three times with five six-sided dice?

Strategy 1 (Use probability and combinations combined):

Probability of rolling 3 die of equal value, assigning them a number from 1-6 and arranging:
(⅙)^3*6=1/36*3C5=5/18;
Probability of rolling 2 die that are not equal to the 3: (⅚)^2=25/36;
Multiply the two steps: 125/648;

Strategy 2 (Use only combinations/counting):

Choose three out of the five positions for the equal numbers: 3C5;
Choose a number out of the 6 possible numbers for the equal numbers: 6;
Choose two numbers that are not equal to the equal numbers: 5*5;

Multiply Steps: 3C5*6*5*5;
Denominator: 6^5;
(3C5*6*5*5)/6^5=125/648;

Dice rolls (easy + Independent): What is the probability of rolling three six-sided dice, and getting a different number on each die?

Strategy 1: Using Probability

1: You choose any number out of 6; 6/6
2: You choose any number out of 6 excluding the previous; 5/6
3. You choose any number out of 6 excluding the previous two; 4/6

Multiply the three steps:

5/6 * 4/6=5/9;

Strategy 2: Using Combinations

There are a total of 6^3 different combinations of 3 numbers taking into account order;
Of these, there are 3A6 different ways to choose 3 numbers out of 6 with order;

Thus, the probability is 3A6/6^3 = 5/9

Dice rolls (Using symmetry principle + Independent): Two standard six-sided dice are tossed. One die is red and the other die is blue. What is the probability that the number appearing on the red die is greater than the number appearing on the blue die?

Solution 1: Using Probability + Counting:

If the number on the blue die is 1 then there are 6-2+1=5 possibilities for the red dice;
b=2; r=6-3+1=4;
b=3; r=6-4+1=3;
b=4; r=2;
b=5 r=1;

Sum of possibilities for r = 15;
Probability for a single case = ⅙*⅙ = 1/36;
Thus total probability = 15*1/36=**15/36**;

Solution 2: Using symmetry principle:

Here we can see that the P(r>b)=P(b>r) given that both dice are equal.
There only case that isn't account here is P(r=b);
Thus, to get P(r>b)=(1-P(r=b))/2;
(1-6/36)/2=**15/36.**

The following problem types generally require dependent probability. It is important to note that for many problems in this category, applying a combinations approach is very difficult and sometimes not possible.

<u>Taking items/numbers/etc. out of groups (Dependent Probability)</u>

Problem 1 (easy + dependent): A jar contains 30 red marbles, 12 yellow marbles, 8 green marbles and 5 blue marbles. You draw and replace marbles 3 times. What is the probability you draw 1 Red, 1 Yellow, and 1 Blue?

Solution: Use dependent probability combined with combinations to solve:

There are 30/55 ways to pick red, 12/55 ways to pick yellow, and 5/55 ways to pick blue. We also need to consider arrangements which can be made in 3! ways; Finding the product, (30*12*55*6)/55^3=~0.06491

Problem 2 (easy + dependent): A jar contains 30 red marbles, 12 yellow marbles, 8 green marbles and 5 blue marbles. You draw and replace marbles 3 times. What is the probability the third marble is the first red marble?

Solution 1: Use dependent probability to solve;

Instead of going through all the distinct cases, we can generalize and say that the probability that the marble is not red is 55-30/55=25/55; We will choose non-red marbles twice, before choosing a red marble. The probability of getting a red marble is 30/55; Thus, the probability is 25/55*25/55*30/55; With order already considered in the 25/55*25/55 section of the expression; The probability is 150/1331;

Solution 2: Using combinations:

We can represent the question with a diagram: N,N,R; There are 25 options for N, and 30 for R; In total, there are 55^3 ways to pick 3 marbles with replacement out of 55. Thus, the probability is: (25*25*30)/55^3=150/1331

Problem 3:

Select 3 marbles at random from a jar with 6-blue, 4-green, and 3-red. What is probability that A.All are green? B.What is probability of selecting 2-blue and 1-red? C.What is probability of selecting in exact order 1 blue, 1 green, and 1 red?

A: Solution 1 (using dependent probability): 4/13*3/13*2/13 = 24/143; Solution 2 (using combinations): 3C4/3C13=24/143

B: Solution (using dependent probability): 6/13*5/12*3/11*3=45/286;

C: Solution (using dependent probability): 6/13*5/12*3/11=6/143;

Problem 4: A bag contains 15 yellow marbles, 10 red marbles, and 5 white marbles. The marbles are identical except for their colors. What is the probability that when three marbles are selected without replacement that one of them is white?

Solution (using dependent probability): Represent the problem with a diagram, and solve using probability;

Diagram: N,N,W; Where N = not white; W = white;

The probability of selecting a non-white marble first is 25/30;
The probability of selecting a non-white marble second is 24/29;
The probability of selecting a white marble third is 5/28;

When taking into account arrangements, it is not hard to see that all arrangements are equal; thus, we can simply multiply the solution of one arrangement by the total number arrangements; There are 3 arrangements total;

25/30*24/29*5/28*3=75/203

Problem 5: Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Solution 1: Using combinations

There are a total of 5C52 ways of picking 5 cards out of 52, not taking into account order.
There are a total of 3C4 ways of picking three aces and a total of 2C4 ways of picking kings. Not that the order in which we pick these items is not relevant because the denominator is not taking into account order.

Thus, 3C4 * 2C4 / 5C52 is the probability;

Solution 2: Using dependent probability combined with combinations;

The probability of picking three aces is 4/52*3/51*2/50;
The probability of picking two kings after the aces is 4/49*3/48;
This method does take into account order, thus we must take into account the arrangements; There are 2C5 ways of arranging the items.
Thus, the probability is 4/52*3/51*2/50*4/49*4/48*2C5 = ;

Note that both expressions are equivalent.

Problem 6: e.g. **4 friends** (Alex, Blake, Chris and Dusty) each choose a random number between 1 and 5. What is the chance that any of them chose the same number?

Solution 1 (efficient): Using complementary method

When the word at least is used, then using the complementary method may be an efficient solution. This problem highlights the possible efficiency of the method. Instead of going through all of the cases, we can simply determine the probability that all of them choose a different number:

5/5*4/5*3/5*2/5 =24/125; Now we simply find the complement: 1-24/125=101/125;

Solution 2: Discussing in cases and using combinations:

4: 5*4C4=5;
1,3: 3C4*5*4
1,1,2: 2C4*5*1C2*4*3/2;
2,2: 2C4*5*2C2*4/2;

Total = 505;
Denominator = 5^4=625;
Probability: 101/125;

Solution 3: Discussing in cases and using probability while considering arrangements:

4: 5/5*⅕*⅕*⅕=1/125*4C4=1/125;
3,1: 5/5*⅕*⅕*⅘=4/125;
2,1,1: 5/5*4/5*⅗*⅕=12/125*2C3*1C2=72/125;
2,2: 5/5*⅘*⅕*⅕=4/125*2C4=24/125

Problem 7: A box contains 2 pennies, 4 nickels, and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of coins drawn is at least 50 cents?

Solution: (Combinations):

The total number of ways to select coins is 4C12 (keep in mind that collecting coins like this considers each coin to be distinguishable)

There are only three cases in which the total value is greater than or equal to 50 cents:

(1): 4 dimes, 2 nickels => 4C6*2C4
(2): 5 dimes, Anything except a dime => 5C6*1C6
(3): 6 dimes => 6C6

Total=90+36+1=127;

Probability of the value of coins drawn being at least 50 cents is 127/924

Problem 8:

Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

Solution 1: Using Probaility1/Probability2 Method

Note that we are trying to find the conditional probability $P(A|B) = \frac{P(A\cap B)}{P(B)}$ where A is the 4 coins being genuine and B is the sum of the weight of the coins being equal. The only possibilities for B are (g is abbreviated for genuine and c is abbreviated for counterfeit, and the pairs are the first and last two in the quadruple) $(g,g,g,g),(g,c,g,c),(g,c,c,g),(c,g,g,c),(c,g,c,g)$. We see that $A\cap B$ happens with probability $\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}=\frac{1}{3}$, and B happens with probability $\frac{1}{3}+4\times\left(\frac{8}{10}\times\frac{2}{9}\times\frac{7}{8}\times\frac{1}{7}\right)=\frac{19}{45}$, hence $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{3}}{\frac{19}{45}}=\boxed{\frac{15}{19}\ (\textbf{D})}$.

Solution 2: Assuming that the identical coins are actually not identical

If we pick 4 indistinguishable real coins from the set of 8 real coins, there are $\binom{8}{4}$ ways to pick the coins. If we then place the coins in four distinguishable slots on the scale, there are $4!$ ways to arrange them, giving $4!\cdot\binom{8}{4}$ ways to choose and place 8 real coins. This gives 1680 desirable combinations.

If we pick 2 real coins and 2 fake coins, there are $\binom{8}{2}\binom{2}{2}$ ways to choose the coins. There are 4 choices for the first slot on the left side of the scale. Whichever type of coin is placed in that first slot, there are 2 choices for the second slot on the left side of the scale, since it must be of the opposite type of coin. There are 2 choices for the first slot on the right side of the scale, and only 1 choice for the last slot on the right side.

Thus, there are $4\cdot2\cdot2\cdot1=16$ ways to arrange the coins, and $\binom{8}{2}\binom{2}{2}=28$ sets of possible coins, for a total of $16\cdot28=448$ combinations that are legal, yet undesirable.

The overall probability is thus $\frac{1680}{1680+448}=\boxed{\frac{15}{19}\ (\textbf{D})}$.

Solution 3: Regular Combinations

WLOG, allow for all the coins to be distinguishable. We split this up into cases. Case 1 being the weight of 2 genuine coins together and Case 2 being the weight of 1 genuine coin and 1 counterfeit coin.

Case 1: All Genuine coins chosen. This happens in $\frac{\binom{8}{2}\cdot\binom{6}{2}}{2}=210$ ways

Case 2: Genuine coin and Counterfeit coin both chosen. This happens in $8\cdot7=56$ ways.

Hence, the answer is $\boxed{\frac{15}{19}\ (\textbf{D})}$.

Problem 10:

Bernardo randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

$\textbf{(A)}\ \frac{47}{72}\qquad\textbf{(B)}\ \frac{37}{56}\qquad\textbf{(C)}\ \frac{2}{3}\qquad\textbf{(D)}\ \frac{49}{72}\qquad\textbf{(E)}\ \frac{39}{56}$

Here, there is a clear similarity between the two sets: The first is the samae as the second except with an added “9”
We can use case work to simplify this:

1. The first person takes the nine: In this case the first person always wins
    a. P(f9)=⅓
2. The first person doesn't take the nine. Now, we just need to calculate the probability that they both take the same numbers:
    a. P(f=s)= 1/56; Thus, the probability that the first person wins and doesn't take a 9 is (1-1/56)/2=55/112; Final prob = ⅔*55/122

Thus, the total probability = 2*55/112*3 + 122/3*122 = 37/56

<u>Circle Probability:</u>

Problem 1:

Problem 22

Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

$\textbf{(A)} \frac{47}{256} \qquad \textbf{(B)} \frac{3}{16} \qquad \textbf{(C)} \frac{49}{256} \qquad \textbf{(D)} \frac{25}{128} \qquad \textbf{(E)} \frac{51}{256}$

We can separate the problem into the following 5 cases:

1. 0 people stand up: 1 * ½^8
2. 1 person stands up: 8*½^8
3. 2 people stand up: ½^8-(2C8-8)
4. 3 people stand up: (use the picking strategy) => 1C8 * (1C3*1C2+1C2*1C3) = 96. Now, we divide by 6 since we don't care which person stands up: 96/6=16*½^8
5. 4 people stand up: 2*1.2^8

<u>Positive or Negative Tests:</u>

Problem 1: Suppose that one of every 500 people in a certain population has a particular disease, which displays no symptoms. A blood test is available for screening for this disease. For a person who has this disease, the test always turns out positive. For a person who does not have the disease, however, there is a 2% false positive rate--in other words, for such people, 98% of the time the test will turn out negative, but 2% of the time the test will turn out positive and will incorrectly indicate that the person has the disease. Let p be the probability that a person who is chosen at random from this population and gets a positive test result actually has the disease. Which of the following is closest to p?

Solution:

This question can also be solved by using a $\frac{probability1}{probability2}$ solution.

If the test is positive there are two cases: (1) The person was true-positive (2) The person was false-positive

For the first case, the probability of the person being true-positive is $\frac{1}{500}$.

For the second case, the probability of the person being false-positive is $\frac{499}{500} \cdot \frac{1}{50}$

The probability of the test being positive is the sum of these values, $\frac{549}{25000}$

The final probability will be in the form: $\frac{P(truepositive)}{P(positive)}$ Plugging in the values we get: $\frac{\frac{1}{500}}{\frac{549}{25000}} = \frac{50}{549}$

Expected Value:

Formulas:

1. E(event1-event2) = E(event1)-E(event2)

C3. Yana and Zahid are playing a game. Yana rolls her pair of fair six-sided dice and draws a rectangle whose length and width are the two numbers she rolled. Zahid rolls his pair of fair six-sided dice, and draws a square with side length according to the rule specified below.

a. Suppose that Zahid always uses the number from the first of his two dice as the side length of his square, and ignores the second. Whose shape has the larger average area, and by how much?

b. Suppose now that Zahid draws a square with the side length equal to the minimum of his two dice results. What is the probability that Yana's and Zahid's shapes will have the same area?

c. Suppose once again that Zahid draws a square with the side length equal to the minimum of his two dice results. Let $D = \text{Area}_{\text{Yana}} - \text{Area}_{\text{Zahid}}$ be the difference between the area of Yana's figure and the area of Zahid's figure. Find the expected value of D.

(a) We use the basic principles of expected value to calculate the expected value for the two cases.

Zahid area expected value = (1^2+2^2+3^2+4^2+5^2+6^2)/6 = **91/6** *(all areas occur in equal frequencies which is why they are weighted equally and dived by 6)*

Yana area expected value = (1+2+3+4+5+6)(1+2+3+4+5+6)/36 = **49/4**

Thus, ANS = **35/12**

(b) We must calculate the weights that were introduced into Zahid's sum.

If zahid gets:

1. 11 possibilities (1 and any other numbers) => 1
2. 9 possibilities => 4
3. 7 possibilities => 9
4. 5 possibilities => 16
5. 3 possibilities => 25
6. 1 possibilities => 36

If yana gets:

1. 1 possibilities => 1
2. 3 possibilities => 4
3. 1 possibilities => 9
4. 1 possibilities => 16

5. 1 possibilities => 25
6. 1 possibilities => 36

9/36*3/36+1/36^2*(11+7+5+3+1) = 54/36^2 = **1/24**

(c)

From (a), we know the expected value of Yanna's area is 49/4. We also know that E(S1-S2)=E(S1)-E(S2)
Thus, we must find the expected value of Zahid's area given the new condition to solve this problem.

(11+36+63+80+75+36)/36 = 301/36

E(s1)=49/4; E(s2)=301/36
E(s1)-E(s2) = ANS = 140/36 = **35/9**

Combinations and Basic Counting:

Types of Problems:

1. Counting Problems
   a. Regular / Basic Counting -> ex. How many ways are there to arrange n objects between n seats?
      i. Case Work
      ii. Complenentary Counting
   b. Unique Counting -> ex. How many ways are there to arrange n keys on a keychain?
2. Tournament / Handshake Problems
3. Pair Problems (Counting)
4. Circle Combinatorics Problem

Problem 1: We are trying to divide 5 European countries and 5 African countries into 5 groups of 2 each. How many ways are there to do this under the restriction that at least one group must have only European countries?

Solution: Using complementary counting

We can first take into account the total number of ways to divide the groups disregarding the given condition. The total number of ways is (2C10*2C8*2C6*2C4*2C2)/5!. We divide by 5! because the order by which we choose the groups is not relevant. Next we can count the number of ways to create 5 groups, each consisting of one European and one African country. 5! Is the number of ways to choose the groups with one European and African country in each. Thus, the total number of ways to choose the groups under the given restriction is 945-120=825.

Problem 2: Find the number of rectangles in a 12 by 10 chess board.

Solution: Using basic combinatorics

We can view the problem as though we are picking 2 lines out of the 11 horizontal ones and then picking 2 lines out of the 13 vertical ones. The four intersection points of these lines will create a rectangle.

2C11*2C13=4290

Problem 3: There are two distinct boxes, 10 identical red balls, 10 identical yellow balls, and 10 identical blue balls. How many ways are there to sort the 30 balls into the two boxes so that each box has 15?

We can translate this problem statement into the following algebraic expression: a+b+c=15. (where 0<=a,b,c<=10). Ignoring the given condition, there are 2C17=136 ways to arrange values in the expression. Now, counting the number of ways in which 10<a,b,c<=15:

Let a=11: b+c=4; => 5 cases
Let a=12: b+c=3; => 4 cases
…
Let a=15: b+c=0; => 1 case

There are a total of 15 cases when a is invalid. There are three variables total, thus 15*3=45 invalid cases.

Solution=136-45=91 cases

Problem 4: Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \leq i \leq 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

We can solve this problem using casework:

We can start with a1=1;
If this is true, then a2=2, a3=3, a4=4…
Thus, there is only one case in which a1=1.

Now a1=2;
If this is true, then a2=1,3,a3=1,4,a4=1,5 … a10=1
If any term is equal to one then we know that only one case will remain such that we only need to count the number of cases that contain one. (all cases) 9 cases.

Now a1=3;
If this is true, then a2=2,4,a3=1,5,a4=1,6.
Similar to the previous case, we need to count the number of cases that end in 1. However, 1 must come after 2 so we are essentially picking two positions out of 9 and placing 2 in the first position and 1 in the second. Thus, 2C9 cases

Now a pattern has made itself evident: 0C9+1C9+2C9+...
This pattern continues until 9C9. Thus, the total is 512

Problem 5:

A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated $90\,^{\circ}$ clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

Here, we use a combination of PEI (essentially just complementary logic) and basic combi;
To start, the center must always be black. Thus, the probability, by default, is in the form ½*n.

Now, let's look at the vertices of the square:

1. If a corner and its diagonal is shaded, then all the corners will eventually become shaded

The probability of at least one of the two vertex pairs being shaded is (½*½)+(½+½)-(½*½*½*½)=7/16 (A+B-AUB)
By symmetry the same is true for the sides: 7/16
Total Probability = (7/16)^2*½ = 49/512

<u>Tournament / Handshake Problems:</u>

Problem 22

A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

$\textbf{(A)}\ 385 \qquad \textbf{(B)}\ 665 \qquad \textbf{(C)}\ 945 \qquad \textbf{(D)}\ 1140 \qquad \textbf{(E)}\ 1330$

Solution

<u>Circle Combinations</u>

Problem 21

Ten chairs are evenly spaced around a round table and numbered clockwise from 1 through 10. Five married couples are to sit in the chairs with men and women alternating, and no one is to sit either next to or across from his/her spouse. How many seating arrangements are possible?

$\mathrm{(A)}\ 240 \qquad \mathrm{(B)}\ 360 \qquad \mathrm{(C)}\ 480 \qquad \mathrm{(D)}\ 540 \qquad \mathrm{(E)}\ 720$

There are 10 choices for the first man, then 4 for the second, then 3, then 2, then 1.

Thus, the combinations for the men is 10*4*3*2*1=240
Then, there are only two possible arrangements for the women that are valid;
Thus, the final probability is 2*240=480.

<u>Distinguishability Problems:</u>

<u>Types of Counting:</u>

1. Distinguishability + Basic Counting Problems
2. Implementing Counting Strategies

<u>Counting Strategies:</u>

1. Creating an arithmetic sequence and using formulas
2. Using combinations / symmetry principle (when a<b)
3. Using Distinguishability Formula / Counting Methods

<u>Distinguishability Formulas:</u>

1. When Groups Distinguishable and Items are Indistinguishable: Total=(m-1)C(n+m-1) (>=0)

2. When Groups Distinguishable and Items are Distinguishable: Total=m^n (>=1)

Distinguishability Problems + Basic Counting Problems:

Problem 1: Seven different prizes are to be distributed among three contest winners such that each winner receives at least one prize and each of the prizes goes to one of the three winners. In how many different ways can the prizes be distributed among the three winners?

We can divide the problem into different cases in which the sum of 3 groups is 7;

(1,1,5)=1C7*1C6*3=126
(1,2,4)=1C7*2C6*6=630
(1,3,3)=1C7*3C6*3=420
(2,2,3)=2C7*2C5*3=630

Sum=1806

Problem 2: How many ways are there to put 4 distinguishable balls into 2 indistinguishable boxes?

Method 1: Overcounting by assuming that both the balls and boxes are distinguishable

If both the balls and boxes are distinguishable then we can apply the formula: m^n;
2^4=16;

Given that we are counting for each rearrangement of the boxes we can simply divide by two, resulting in 16/2=8;

Method 2: Using counting

The three cases are: (0,4), (1,3), (2,2); Yielding 4C4+1C4+(2C4)/2=8;

Problem 3: How many ways are there to put 4 balls in 3 boxes if the balls are distinguishable but the boxes are not?

Here are the 4 cases: (4,0,0),(3,1,0),(2,2,0),(2,1,1): These cases yield a collective result of 4C4 + 1C4 + 2C4/2 + 2C4 = 14

Problem 4: How many ordered non-negative integer solutions (a, b, c, d) are there to the equation a + b + c + d = 10?

We are distributing 10 1's among four distinct groups and any group may receive 0 items.
Thus, we can implement the formula: 3C13=286;

Problem 5: You want to distribute 7 indistinguishable candies to 4 kids. If every kid must receive at least one candy, in how many ways can you do this?

We can first assign 4 candies to the four kids (we do not need to care about the order or specific candies that we give out because they are all indistinguishable) and we are left with 3 candies to assign to the four kids where each kid may receive 0 candies. Now we can apply the formula: 3C(3+4-1)=3C6=20

Problem 6:

For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

(a+b+c+d+1)^N = (a+b+c+d+1)(a+b+c+d+1)(a+b+c+d+1)(a+b+c+d+1)...

Essentially, we select which number we want to contribute to the total product of each term by selecting one number in each bracketed term.

Each term, must be represented as follows: a^x * b^y * c^z * d^q * 1^p (where x,y,z,q>0 and x+y+z+q+p = N)

This structure, quite frankly, screams out “STARS AND BARS”

Applying, the classic distinuighsbility formula: N+4-4C4 = NC4 = 1001;

Thus, we can solve for N

The most important “take away” is:

1. Recognizing that (a+b+c+..)^n = (a+b+c+..)(a+b+c+..)(a+b+c+..) * … and that we can implement combinatorics along with that perspective to trivialize the problem.

Problem 7:

Problem 22

Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

$\textbf{(A)}\ 1930 \qquad \textbf{(B)}\ 1931 \qquad \textbf{(C)}\ 1932 \qquad \textbf{(D)}\ 1933 \qquad \textbf{(E)}\ 1934$

We know that, with no restrictions, we can simply say that the number of distributions is 3^7.

The number of ways in which the red bag has 0 candies is 2^7. The same is true for the blue bag. Thus total = 2^8
The number of ways in which both the red and blue bag have 0 candies is 1. Thus, ANS = 3^7-2^8+1 = ©

Implementing Counting Strategies:

Problem 1 (Complex Contest Counting + Statistics Problem): Consider positive integers a<=b<=c<=d<=e. There are N lists a,b,c,d,e with a mean of 2023 and a median of 2023, in which the integer 2023 appears more than once, and in which no other integer appears more than once. What is the sum of the digits of N?

If 2023 exists more than 2 times in the list N, then the list will consist of all 2023's because the median must be 2023. Thus, the only case to consider is when 2023 occurs twice.

<u>Case 1: (c=2023, b=2023)</u>

In this case, $a+d+e = 2023 * 3$, and $a<2023<d<e$

The minimum value for d is 2024 (+1). So e can be from 2025 (+2) to 4044 (+2021). [2020]
The next value for d is 2025 (+2). So e can be from 2026 (+3) to 4043 (+2020) [2018]
…
The maximum value for d occurs when a is minimized (a=1 ‖ -2022). In this case, the maximum value for d is 1010 and e will be 1012 or 1011. [2]

Counting the arithmetic sequence we get: 1011*1010.

<u>Case 2: (c=2023, d=2023)</u>

In this case, $a+b+e=2023*3$, and $a < b < 2023 < e$

Counting Method 1: (Arithmetic sequence counting)

The minimum value for a is 1 and b can then span from 2 to 2022 [2021 cases]
The next value for a is 2 and b can then span from 3 to 2022 [2019 cases]
….
The maximum value for a is 2021 and b can only be 2022 [1 case]

Counting the arithmetic sequence: 2021*1011

Counting Method 2: (Combinatorics)

We are essentially choosing two distinct numbers from the set of numbers [1,2,3…2022] and placing them in increasing order. We can solve for the number of cases with 2C2022 = 2021*1011

<u>Case 3: (everything is 2023) Only 1 occurrence of this case. (2023,2023,2023,2023,2023)</u>

<u>Solution</u>: Case 1 + Case 2 + Case 3 = 1011*1010 + 2021 * 1011 + 1 = (A sum of 22)

Problem 2:

A group of n friends wrote a math contest consisting of eight short-answer problems $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$, and four full-solution problems F_1, F_2, F_3, F_4. Each person in the group correctly solved exactly 11 of the 12 problems. We create an 8 x 4 table. Inside the square located in the i^{th} row and j^{th} column, we write down the number of people who correctly solved both problem S_i and problem F_j. If the 32 entries in the table sum to 256, what is the value of n?

	F_1	F_2	F_3	F_4
S_1				
S_2				
S_3				
S_4				
S_5				
S_6				
S_7				
S_8				

To solve this, we recognize the fact that each person doesn't solve either a single Fi or a single Si. If the person doesn't solve an Fi, then their total score is 24; If the person doesn't solve an Si, then their total score is 28;

Thus, the sum of all the scores scored by all the people can be represented as 24x+28y=256;

Now, we can test through all values of (x,y) starting from x=10 down to x=0 and find the valid cases and corresponding values of x,y; This helps us find n because n=x+y

Shortest Path Problems:

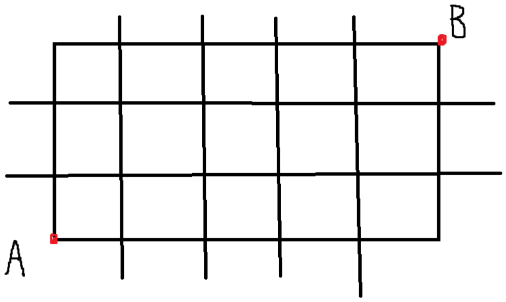
Typical Types of Problems:

1. **2d shape:**
    - Basic point to point
    - Going through a point
    - Not going through a point
    - Not one direction

2. **3d shape:**
    - Basic point to point
    - Going through a point
    - Not one direction
3. **Recursive counting paths:**
4. **Sequence Paths**

2d-shape:

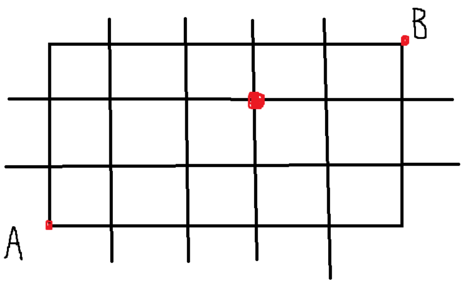
Problem 1: How many shortest paths are there from A to B?



Strategy:
Step 1: Find an example of the shortest path (Easiest is going along the outer sides)
Step 2: Divide route into different directions (up and down in 2d shapes)
Step 3: Use combination formula to solve

An example of a shortest route consists of up = 3 and right: = 5;
Solution: min(up,right)C(up+down) = $\frac{8*7*6}{3!}$=56;

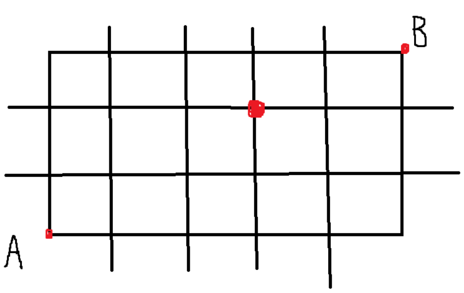
Problem 2: How many shortest paths passing the indicated intersection?



Strategy: Divide into groups and use the same logic used in the question prior to determine the individual number of paths in each group. After, multiply the results

Section 1: Up = 2, Right =3 (Number of paths = 2C5 = 10)
Section 2: Up = 1, Right 2 (Number of paths = 1C2=2)
Solution: 10 * 2 = 20;

Problem 3: How many paths DO NOT pass through the indicated intersection?



Strategy: Find the total number of shortest paths going from A to B directly and subtract it by the number of shortest paths passing through the intersection;

Total Section: Right 5, Up 3 (3C8=56)
Section 1: 2C5=10;
Section 2: 1C2=2;
Paths passing through intersection = 2 *10 =20;
Paths not passing through intersection = 56 -20 = 36

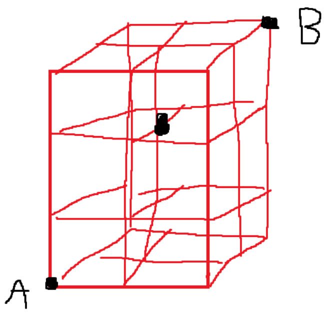
Problem 4: How many paths if it is not allowed to move up more than one time in a row?

Strategy: Use the insertion method to determine the result;

^_^_^_^_^_^ You are inserting the 3 possible ups among the 6 possible positions; 3C6=20;

3d shapes:

Diagram for all the problems



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Problem 1: How many shortest paths from A to B?

Problem Strategy:
Determine one of the many shortest paths and divide it into different directions
After, use combinations to determine the solution.

Can be divided into 3 width, 2 length, and 2 depth
Solution: 3C7*2C4*2C2=210

Problem 2: How many paths from A to B you can;t move right more than once at a time?

Strategy: Use insertion method

^_^_^_^_^_^ There are 2 lengths and 3 ups being combined in the _ category; we can distribute them such that the number of arrangements = 2C5=10; We are choose 2 positions out of the 6 to place a right in such that the number of arrangements = 2C6 = 15; Now we multiply both steps to get 15 * 10 = 150;

Problem 3: How many paths that pass through the intersection?

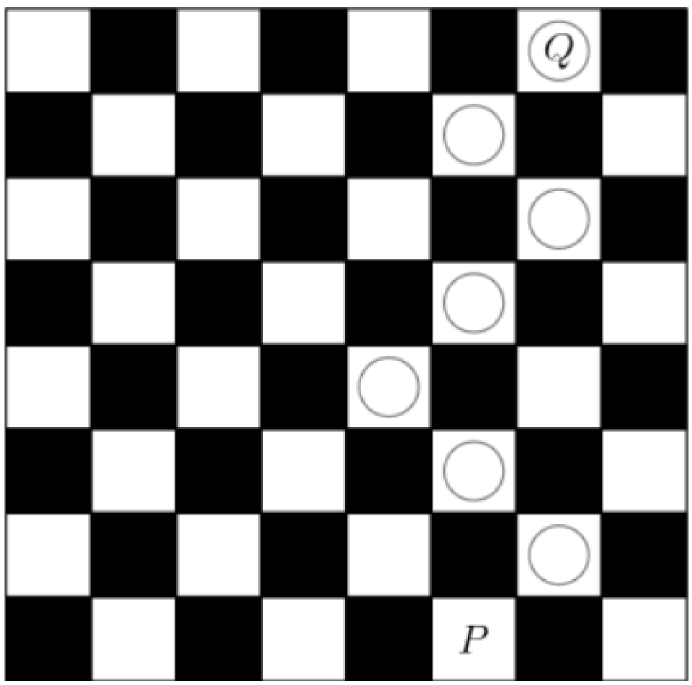
Strategy: Divide into 2 separate 3d shapes and solve them individually; After multiply their results;
Shape 1: 1C4 * 2C3 = 12;
Shape 2: 1C3*1C2 = 6;

Total: 6 * 12 = 72;

Recursion paths: (Typically Diagonals):

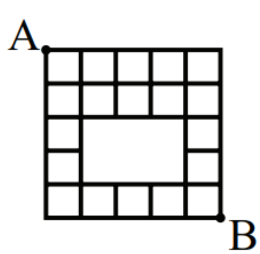
Strategy: Determine the number of paths, starting from the most basic path, and recursively moving along

Problem 1: A game board consists of 64 squares that alternate in color between black and white. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at $P.$ A step consists of moving the marker onto one of the adjoining white squares in the row above. How many 7-step paths are there from P to $Q?$ (The figure shows a sample path.)



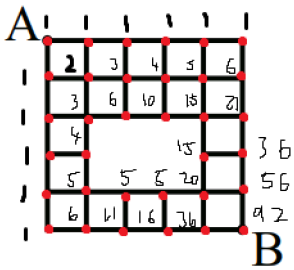
Strategy:
Starting at Q, solve for the number of paths to the starting squares and slowly move down in a recursive fashion
Solution = 28 paths;

In this figure, how many paths go to the right and down along connected segments from point A to point B?



Problem 2:

Strategy: Determine the number of paths that lead to each node recursively, eventually leading to the target node, B.



92 paths from A to B (only going right and up)

Sequence Paths:

In the grid shown, how many ways are there to spell the word "QUEUE" by moving one square at a time either horizontally or vertically, and provided squares may be revisited?

E	U	E	U	E
U	E	U	E	U
E	U	Q	U	E
U	E	U	E	U
E	U	E	U	E

Problem 1:

Strategy: Create a tree and solve <u>(take into account that letters might not have the same environment)</u>

Q => U: 4 U's with identical environments
U => E: 2 E's with identical environments, 1 without
E1 => U 4 U's with identical environments; E2=> 3 U's with identical environments
U1 => 3 E's with identical environments; U2=> 3 E's with identical environments
4*(2*4*3+1*3*3)=(24+9)*4=**132**;

Problem 2:

Todo:

1. Math Challengers Summarization (Due Date Jan. 5th):
    -> 2018: Blitz (26); Bullseye (4,7,8,12)
    -> 2017: Blitz (21,24,25,26); Bullseye (3,4,5,7,8,12);
    -> 2016: Blitz (Oct., 25,26); Bullseye (4,11,12)

Additional Practice Resources:

<u>Combinatorics</u>

- Probability: https://brilliant.org/site_media/files/pdfs/competition-fundamentals/probability-practice-quiz-2.pdf

- https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Dice_Problems.pdf
- Counting:
  https://people.csail.mit.edu/ddeford/Crossroads_pdfs/counting_intro.pdf
  https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Warmup_Counting.pdf
- Solving for combinations:

Word Problems

Work Output

- Traveling Word Problems
- Working Word Problems

Ratios

- Mixture Word Problems
- Age Problems

Money

- Store-cost Problems
- Price/Pound Problems
- Interest Problems

Quadratic Formula + Bases

- Implementation of Quadratics
- Base Problems:
- https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Bases_arithmetic.pdf

Sequences

- Arithmetic Sequence (Common Problems)
- Geometric Sequence (Common Problems)
- Repeating Group (Common Problems)
- Difficult Contest Problems

Geometry

- Solving for area (Different Question Types)
- Using Pythagorean Theorem
- Volume Problems
- Angles Problems
- Similarity + Congruence Problems
- Plane Geometry Problems
- Trigonometric Problems
- Circle Geometry
- Path (Shortest, Longest, etc.)
- Difficult Contest Problems

Number Theory

- Modular Arithmetic (Typical Problems):
  https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Date_Warmups.pdf
- Working with Exponents
- Magic Squares/Triangles
- Working with GCF and LCM
- Sets
- Difficult Contest Problems

Additional Practice Resource: ** = Very Important

1. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Word_Problems.pdf
2. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/New_Warmups.pdf
3. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/More_More_Warmups.pdf

4. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/further_warmups.pdf
5. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/More_Warmups.pdf
6. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Warmups.pdf
7. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/warmup4.pdf
8. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/warmup3.pdf
9. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Practice_Team2.pdf
10. https://people.csail.mit.edu/ddeford/Crossroads_pdfs/Warm_up.pdf
11. https://artofproblemsolving.com/wiki/index.php/AMC_10_Problems_and_Solutions **
12. https://www.egbc.ca/getmedia/5cfa019a-f50b-4a12-97b0-e8ebf020ce71/Quest-Archive.pdf.aspx **

Content Dump:

Suppose that N can be written in base 6 as 531340_6 and in base 8 as 124154_8. In base 10, what is the remainder when N is divided by 210?

Your First Answer: d
Your Second Answer: give up
Solution:
The prime factorization of $210 = 2 \cdot 3 \cdot 5 \cdot 7$. By the Chinese Remainder Theorem, it suffices to find the residues of N modulo 5, 6, and 7. Since the units digit of N in base 6 is equal to 0, it follows that N is divisible by 6. Also, we note that N is congruent modulo $b-1$ to the sum of its base b digits. Indeed, if N can be represented as $(\overline{a_k a_{k-1} \cdots a_0})_b$, then

$$\begin{aligned} N &\equiv a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots + a_1 \cdot b + a_0 \\ &\equiv a_k \cdot ((b-1)+1)^k + \cdots + a_1 \cdot ((b-1)+1) + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod{b-1}. \end{aligned}$$

It follows that $N \equiv 5+3+1+3+4+0 \equiv 1 \pmod 5$ and that $N \equiv 1+2+4+1+5+4 \equiv 3 \pmod 7$. By the Chinese Remainder Theorem and inspection, we determine that $N \equiv 31 \pmod{35}$, so that (by the Chinese Remainder Theorem again) $N \equiv \boxed{66} \pmod{210}$.

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Your First Answer: 71
Your Second Answer: give up
Solution:
In $f(x)$, all terms will have a multiple of x except for the constant term, which is the multiple of the four constants $4, 1, 6$, and 9.

Recall (from the Euclidean algorithm) that the greatest common divisor of a and b is the same as the greatest common divisor of a and $a - kb$ where $k, a,$ and b are any integers. Therefore, finding the greatest common divisor of $f(x)$ and x is the same as finding the greatest common divisor of x and the constant term of $f(x)$. Therefore, we want to find

$$\begin{aligned} \gcd\left((3x+4)(7x+1)(13x+6)(2x+9), x\right) &= \gcd\left(4 \cdot 1 \cdot 6 \cdot 9, x\right) \\ &= \gcd\left(216, x\right) \end{aligned}$$

Since 15336 is a multiple of 216, the greatest common divisor of $f(x)$ and x is $\boxed{216}$.

orrect! Way to go!

What is the smallest positive integer n such that $531n \equiv 1067n \pmod{24}$?

Your Answer: 3

Solution:

Recall that, by definition, $531n \equiv 1067n \pmod{24}$ means that $531n - 1067n$ is divisible by 24. In other words,

$$\frac{1067n - 531n}{24} = \frac{536n}{24} = \frac{67n}{3}$$

must be an integer. Since 67 and 3 are relatively prime, n must be a multiple of 3, the smallest of which is $\boxed{3}$.

ou gave up on this problem. Source: Introduction to Counting & Probability Chapter 11

Two fair eight-sided dice have their faces numbered from 1 to 8. What is the expected value of the sum of the rolls of both dice?

Your First Answer: s

Your Second Answer: give up

Solution:

To find the expected value of a double roll, we can simply add the expected values of the individual rolls, giving $4.5 + 4.5 = \boxed{9}$.

$$T_{(r+1)} = {}^{n}C_{r} \times a^{n-r} \times b^{r}$$

Solution

Level 17 Counting & Probability **+98 XP** **Level 8** Intermediate Algebra **+55 XP**

You gave up on this problem. In topic: Binomial Theorem (Counting & Probability) +1 more. Source: AoPS Staff

What is the coefficient of x^3 in the expansion of

$$(x + 2\sqrt{3})^7?$$

Your First Answer: 840sqrt(3)

Your Second Answer: give up

Solution:

By the binomial theorem, this term is

$$\binom{7}{3} x^3 (2\sqrt{3})^4 = 35x^3 \cdot 144 = \boxed{5040} x^3.$$

Correct! Way to go! In topic: Counting with Symmetry (Counting & Probability). Source: Introduction to Counting & Probability Chapter 3

In how many ways can 9 people sit around a round table? (Two seatings are considered the same if one is a rotation of the other.)

Your Answer: 40320

Solution:

There are $9!$ ways to arrange 9 people in a line, however there are 9 identical rotations for each arrangement, so we divide by 9 to get $\dfrac{9!}{9} = 8! = \boxed{40{,}320}$.

What is the minimum number of times you must throw three fair six-sided dice to ensure that the same sum is rolled twice?

Your First Answer: 7
Your Second Answer: give up
Solution:
In the worst-case scenario, every possible sum is rolled before the same sum is rolled again. The minimum possible sum rolled is $3 \cdot 1 = 3$, and the maximum is $3 \cdot 6 = 18$. Every sum in between those two extremes can be created, since the sums are created through adding three of the digits between one and six. Thus, there are $18 - 2 = 16$ possible sums, so the dice must be rolled $\boxed{17}$ times to ensure that the same sum is rolled twice.

What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

Your First Answer: 0
Your Second Answer: 4
Solution:
Since $n!$ contains the product $2 \cdot 5 \cdot 10 = 100$ whenever $n \geq 10$, it suffices to determine the tens digit of

$$7! + 8! + 9! = 7!(1 + 8 + 8 \cdot 9) = 5040(1 + 8 + 72) = 5040 \cdot 81.$$

This is the same as the units digit of $4 \cdot 1$, which is $\boxed{4}$.

Your First Answer: k
Your Second Answer: give up
Solution:
From Pascal's identity $\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$.

Therefore, we have $\binom{20}{11}+\binom{20}{10}=\binom{21}{11}$, so $n=11$.

We know that $\binom{21}{11}=\binom{21}{21-11}=\binom{21}{10}$.

We use Pascal's identity again to get $\binom{20}{9}+\binom{20}{10}=\binom{21}{10}$, so $n=9$.

There are two values for n, 9 and 11, so the sum is $9+11=\boxed{20}$.

Your Answer: 193/512
Solution:
There are $2^{10}=1024$ possible outcomes of the 10 coin flips. The probability that we flip at least 6 heads is equal to the probability that we flip at least 6 tails, by symmetry. Let's call this probability p. The only other possibility is that we flip exactly 5 heads and 5 tails, for which the probability is $\frac{\binom{10}{5}}{2^{10}}=\frac{252}{1024}=\frac{63}{256}$. Therefore, $\frac{63}{256}+2p=1$, giving

$$p=\frac{1}{2}\left(1-\frac{63}{256}\right)=\boxed{\frac{193}{512}}$$

What is the greatest product obtainable from two integers whose sum is 246?

Your Answer: 15129
Solution:
Let the two integers be x and $246-x$. The product which needs to be maximized is $(x)(246-x)=246x-x^2$. Now we complete the square:

$$\begin{aligned}-(x^2-246x)&=-(x^2-246x+15129-15129)\\&=-(x-123)^2+15129\end{aligned}$$

Since the square of a real number is always non-negative, $-(x-123)^2\le 0$. Therefore, the expression is maximized when $x-123=0$, so $x=123$. Therefore, the greatest product obtainable is $-(123-123)^2+15129=\boxed{15129}$.